

THRESHOLD SOLUTIONS FOR THE FOCUSING L^2 -SUPERCRITICAL NLS EQUATIONS

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ABSTRACT. We investigate the L^2 -supercritical and \dot{H}^1 -subcritical nonlinear Schrödinger equation in H^1 . In [6] and [20], the mass-energy quantity $M(Q)^{\frac{1-s_c}{s_c}} E(Q)$ has been shown to be a threshold for the dynamical behavior of solutions of the equation. In the present paper, we study the dynamics at the critical level $M(u)^{\frac{1-s_c}{s_c}} E(u) = M(Q)^{\frac{1-s_c}{s_c}} E(Q)$ and classify the corresponding solutions using modulation theory, non-trivially generalize the results obtained in [9] for the 3D cubic Schrödinger equation.

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1. INTRODUCTION

We consider the following Cauchy problem of a nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u + |u|^{p-1}u = 0, & (x, t) \in \mathbb{R}^N \times \mathbb{R}, \\ u(x, 0) = u_0(x) \in H^1(\mathbb{R}^N). \end{cases} \quad (1.1)$$

It is well known from [4] and [1] that, equation (1.1) is locally well-posed in H^1 . That is for $u_0 \in H^1$, there exist $0 < T \leq \infty$ and a unique solution $u(t) \in C([0, T]; H^1)$ to (1.1). When $T = \infty$, we say that the solution is positively global; while on the other hand, we have $\lim_{t \uparrow T} \|\nabla u(t)\|_2 \rightarrow \infty$ and call that this solution blows up in finite positive time. Solutions of (1.1) admits the following conservation laws in energy space H^1 :

$$\begin{aligned} L^2 - norm : \quad M(u)(t) &\equiv \int |u(x, t)|^2 dx = M(u_0); \\ Energy : \quad E(u)(t) &\equiv \frac{1}{2} \int |\nabla u(x, t)|^2 dx - \frac{1}{p+1} \int |u(x, t)|^{p+1} dx = E(u_0); \\ Momentum : \quad P(u)(t) &\equiv \operatorname{Im} \int \bar{u}(x, t) \nabla u(x, t) dx = P(u_0). \end{aligned}$$

Note that equation (1.1) is invariant under the scaling $u(x, t) \rightarrow \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t)$ which also leaves the homogeneous Sobolev norm \dot{H}^{s_c} invariant with $s_c = \frac{N}{2} - \frac{2}{p-1}$. Other scaling invariant quantities are $\|\nabla u\|_2 \|u\|_2^{\frac{1-s_c}{s_c}}$ and $E(u)M(u)^{\frac{1-s_c}{s_c}}$. It is classical from the conservation of the energy and the L^2 norm that for $s_c < 0$, the equation is subcritical and all H^1

solutions are global and H^1 bounded. The smallest power for which blow up may occur is $p = 1 + \frac{4}{N}$ which is referred to as the L^2 critical case corresponding to $s_c = 0$ (see [5] [13]). The case $0 < s_c < 1$ (equivalent to $1 + \frac{4}{N} < p < 1 + \frac{4}{N-2}$) is called the L^2 supercritical and H^1 subcritical case. In this paper, we are concerning with the case $0 < s_c < 1$.

We say that (q, r) is $\dot{H}^s(\mathbb{R}^N)$ -admissible ($0 \leq s \leq 1$) denoted by $(q, r) \in \Lambda_s$ if

$$\frac{2}{q} + \frac{N}{r} = \frac{N}{2} - s, \quad \frac{2N}{N-2s} < r < \frac{2N}{N-2}.$$

This is associated to the well-known Strichartz's estimates: for any $\varphi \in \dot{H}^s$, $f(x, t) \in L_t^q L_x^r$ and any admissible pair $(q, r), (\gamma, \rho) \in \Lambda_s$, we have

$$\|e^{it\Delta}\varphi\|_{L_t^q L_x^r} \leq C\|\varphi\|_{\dot{H}^s}, \quad \|Gf\|_{L_t^{\gamma'} L_x^{\rho'}} \leq C\|f\|_{L_t^q L_x^r}, \quad (1.2)$$

where $\frac{1}{\rho'} + \frac{1}{\rho} = \frac{1}{\gamma'} + \frac{1}{\gamma} = 1$, and $Gf(t, x) \equiv \int_{t_0}^t e^{i(t-s)\Delta} f(s) ds$. We define the following Strichartz norm

$$\|u\|_{S(\dot{H}^s)} = \sup_{(q,r) \in \Lambda_s} \|u\|_{L_t^q L_x^r}$$

and recall the following properties for the Cauchy problem (1.1), which can be found in [20]:

Proposition 1.1. (*Small initial data*). *Let $\|u_0\|_{\dot{H}^{s_c}} \leq A$, then there exists $\delta_{sd} = \delta_{sd}(A) > 0$ such that if $\|e^{it\Delta}u_0\|_{S(\dot{H}^{s_c})} \leq \delta_{sd}$, then u solving (1.1) is global and*

$$\|u\|_{S(\dot{H}^{s_c})} \leq 2\|e^{it\Delta}u_0\|_{S(\dot{H}^{s_c})}, \quad (1.3)$$

$$\|D^{s_c}u\|_{S(L^2)} \leq 2c\|u_0\|_{\dot{H}^{s_c}}. \quad (1.4)$$

Remark 1.2. Note that by Strichartz's estimates, the hypotheses are satisfied if $\|u_0\|_{\dot{H}^{s_c}} \leq C\delta_{sd}$. Furthermore, by the result obtained by [20], the uniform bound of \dot{H}^{s_c} -norm of the solution u to (1.1) implies $u(t)$ scatters as $t \rightarrow \pm\infty$.

Proposition 1.3. (*Existence of wave operators*). *Suppose that $\psi^+ \in H^1$ and*

$$\frac{1}{2}\|\nabla\psi^+\|_2^2 M(\psi^+)^{\frac{1-s_c}{s_c}} < E(Q)M(Q)^{\frac{1-s_c}{s_c}}. \quad (1.5)$$

Then there exists $v_0 \in H^1$ such that v solves (1.1) with initial data v_0 globally in H^1 with

$$\|\nabla v(t)\|_2 \|v_0\|_2^{\frac{1-s_c}{s_c}} < \|\nabla Q\|_2 \|Q\|_2^{\frac{1-s_c}{s_c}}, \quad M(v) = \|\psi^+\|_2^2, \quad E[v] = \frac{1}{2}\|\nabla\psi^+\|_2^2,$$

and

$$\lim_{t \rightarrow +\infty} \|v(t) - e^{it\Delta}\psi^+\|_{H^1} = 0.$$

Moreover, if $\|e^{it\Delta}\psi^+\|_{S(\dot{H}^{s_c})} \leq \delta_{sd}$, then

$$\|v_0\|_{\dot{H}^{s_c}} \leq 2\|\psi^+\|_{\dot{H}^{s_c}} \quad \text{and} \quad \|v\|_{S(\dot{H}^{s_c})} \leq 2\|e^{it\Delta}\psi^+\|_{S(\dot{H}^{s_c})}.$$

$$\|D^s v\|_{S(L^2)} \leq c\|\psi^+\|_{\dot{H}^s}, \quad 0 \leq s \leq 1.$$

Proposition 1.4. (*long time perturbation theory*). $\forall A \geq 1$, there exists $\epsilon_0 = \epsilon_0(A)$, $c = c(A) \gg 1$ such that if $u = u(x, t) \in H^1$ satisfy

$$iu_t + \Delta u + |u|^{p-1}u = 0.$$

$\tilde{u} = \tilde{u}(x, t) \in H^1$, define

$$e = i\tilde{u}_t + \Delta \tilde{u} + |\tilde{u}|^{p-1}\tilde{u}$$

with $\|\tilde{u}\|_{S(\dot{H}^{s_c})} \leq A$. If

$$\|e\|_{S(\dot{H}^{s_c})} \leq \epsilon_0,$$

$$\|e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0))\|_{S(\dot{H}^{s_c})} \leq \epsilon_0,$$

then

$$\|u\|_{S(\dot{H}^{s_c})} \leq c = c(A) < \infty.$$

For the 3D cubic nonlinear Schrödinger equation with $s_c = \frac{1}{2}$ and $p = 3$, there have been several results on either scattering or blow-up solutions. In [7], [2] and [8], Roudenko and Holmer have shown that $M(Q)E(Q)$ plays an important role in the dynamical behavior of solutions of equation (1.1) with $p = 3$ and $N = 3$. The authors in [20] and [6] extended their results to the general L^2 -supercritical and \dot{H}^1 -subcritical case and showed that $M(Q)^{\frac{1-s_c}{s_c}}E(Q)$ is an threshold for the dynamics in the following sense: Let u be a solution of (1.1) satisfying $M(u)^{\frac{1-s_c}{s_c}}E(u) < M(Q)^{\frac{1-s_c}{s_c}}E(Q)$. Then if $\|\nabla u_0\|_2 \|u_0\|_2^{\frac{1-s_c}{s_c}} < \|\nabla Q\|_2 \|Q\|_2^{\frac{1-s_c}{s_c}}$, we have $T_+ = T_- = \infty$ and $\|u\|_{S(\dot{H}^{s_c})} < \infty$. On the other hand, if $\|\nabla u_0\|_2 \|u_0\|_2^{\frac{1-s_c}{s_c}} > \|\nabla Q\|_2 \|Q\|_2^{\frac{1-s_c}{s_c}}$, then either $u(t)$ blows up in finite forward time, or $u(t)$ is forward global and there exists a time sequence $t_n \rightarrow \infty$ such that $\|\nabla u(t_n)\|_2 \rightarrow \infty$. A similar statement holds for negative time. Our goal in this paper is to give a classification of solutions of the solution of (1.1) with the critical level:

$$M(u)^{\frac{1-s_c}{s_c}}E(u) = M(Q)^{\frac{1-s_c}{s_c}}E(Q) \quad (1.6)$$

extending the very recent results obtained in [9] for the particular case with $p = 3$ and $N = 3$. The idea in this paper follows from Kenig-Merle [3] for the energy-critical NLS.

In this paper we obtain the following results:

Theorem 1.5. *There exist two radial solutions Q^+ and Q^- of (1.1) with initial data $Q_0^\pm \in \cap_{s \in \mathbb{R}} H^s(\mathbb{R}^N)$ and satisfy*

(a) $M(Q^+) = M(Q^-) = M(Q)$, $E(Q^+) = E(Q^-) = E(Q)$, $[0, +\infty)$ is in the domain of the definition of Q^\pm and there exists $e_0 > 0$ such that

$$\|Q^\pm(t) - e^{i(1-s_c)t}Q\|_{H^1} \leq Ce^{-e_0 t}, \quad \forall t \geq 0;$$

(b) $\|\nabla Q_0^-\|_2 < \|\nabla Q\|_2$, Q^- is globally defined and scatters for negative time;

(c) $\|\nabla Q_0^+\|_2 > \|\nabla Q\|_2$, and the negative time of existence of Q^+ is finite.

Theorem 1.6. *Let u be a solution of (1.1) satisfying (1.6).*

- (a) *If $\|\nabla u_0\|_2 \|u_0\|_2^{\frac{1-s_c}{s_c}} < \|\nabla Q\|_2 \|Q\|_2^{\frac{1-s_c}{s_c}}$, then either u scatters or $u = Q^-$ up to the symmetries;*
- (b) *If $\|\nabla u_0\|_2 \|u_0\|_2^{\frac{1-s_c}{s_c}} = \|\nabla Q\|_2 \|Q\|_2^{\frac{1-s_c}{s_c}}$, then $u = e^{i(1-s_c)t}Q$ up to the symmetries;*
- (c) *If $\|\nabla u_0\|_2 \|u_0\|_2^{\frac{1-s_c}{s_c}} > \|\nabla Q\|_2 \|Q\|_2^{\frac{1-s_c}{s_c}}$, and u_0 is radial or of finite variance, then either the interval of existence of u is of finite or $u = Q^+$ up to the symmetries.*

Remark 1.7. Equation (1.1) admits the Galilean invariance: If $u(x, t)$ is a solution of (1.1), then for any $\xi_0 \in \mathbb{R}^N$, $w(x, t) \equiv u(x - \xi_0 t, t) e^{i \frac{\xi_0}{2} \cdot (x - \frac{\xi_0}{2} t)}$ also satisfies the equation (1.1). Recall from the Appendix of [6], taking the Galilean transform with $\xi_0 = -P(u)/M(u)$ into account, we get a solution with zero momentum which is the minimal energy solution v among all Galilean transformations of the solution u of (1.1). Precisely, $M(v) = M(u)$, $E(v) = E(u) - \frac{1}{2} \frac{P(u)^2}{M(u)}$ and $\|v_0\|_2^2 = \|u_0\|_2^2 - \frac{1}{2} \frac{P(u_0)^2}{M(u_0)}$. Applying Theorem 1.6 and the results obtained in [20] and [6] to v , we indeed obtain that

Theorem 1.8. *Let u be a solution of (1.1) satisfying*

$$M(u)^{\frac{1-s_c}{s_c}} E(u) - \frac{1}{2} P(u)^2 \leq M(Q)^{\frac{1-s_c}{s_c}} E(Q).$$

Then,

- (a) *If $\|\nabla u_0\|_2 \|u_0\|_2^{\frac{1-s_c}{s_c}} - P(u)^2 < \|\nabla Q\|_2 \|Q\|_2^{\frac{1-s_c}{s_c}}$, then either u scatters or $u = Q^-$ up to the symmetries;*
- (b) *If $\|\nabla u_0\|_2 \|u_0\|_2^{\frac{1-s_c}{s_c}} - P(u)^2 = \|\nabla Q\|_2 \|Q\|_2^{\frac{1-s_c}{s_c}}$, then $u = e^{i(1-s_c)t}Q$ up to the symmetries;*
- (c) *If $\|\nabla u_0\|_2 \|u_0\|_2^{\frac{1-s_c}{s_c}} - P(u)^2 > \|\nabla Q\|_2 \|Q\|_2^{\frac{1-s_c}{s_c}}$, and u_0 is radial or of finite variance, then either the interval of existence of u is of finite or $u = Q^+$ up to the symmetries.*

The outline of this paper is as follows. In section 2, we recall some properties of the ground state Q and analyze the linearized equation associated to (1.1) near $e^{i(1-s_c)t}Q$. In section 3, we construct a family of approximate solutions using the discrete spectrum of the linearized operator and produce candidates for the special solutions Q^+ and Q^- . Then in section 4, we discuss the modulational stability near Q , which is important for our study of solutions with initial data from part (a) and (c) in Theorem 1.6. This is done in sections 5 and 6 respectively. In section 7, we establish the uniqueness of special solutions by analyzing the linearized equation and finally finish the proof of the classification of solution in the critical level.

This paper is a non-trivial generalization of [9], which deals with the 3D cubic Schrödinger equations. First of all, quite different from the case $p = 3, N = 3$ considered in [9], our p is not an integer when $N \geq 4$, since $1 + \frac{4}{N} < p < 1 + \frac{4}{N-2}$. This mainly brings two difficulties for our study as follows. On the one hand, it is not enough to consider the problem just in the space $C_b(I; H^1)$ as the authors did in [9], where $I \subset \mathbb{R}$ is a time interval. Instead, we should also work on the Strichartz space $L^{\frac{4(p+1)}{N(p-1)}}(I; L^{p+1}(\mathbb{R}^N))$ and use the corresponding Strichartz's estimates associated to the Schrödinger operator $e^{it(\Delta - (1-s_c))}$, which is

just like the classical Strichartz's estimates. On the other hand, the general case require more sophisticated analyzing on the spectral properties of the linearized Schrödinger operators. Moreover, because of the technical difficulties, we cannot directly use the linearized equation near $e^{it}\tilde{Q}$ with \tilde{Q} solving the elliptic equation $-\Delta Q + Q - Q^p = 0$ as the authors did in [9]; while instead, we linearize the equation near $e^{i(1-s_c)t}Q$, where Q solves $-\Delta Q + (1 - s_c)Q - Q^p = 0$.

In this paper, we denote the Sobolev spaces $H^1(\mathbb{R}^N)$ and $W^{m,p}(\mathbb{R}^N)$ as H^1 and $W^{m,p}$ for short, and the L^p norm as $\|\cdot\|_p$. C is denoted variant absolute constants only depending on N and p .

2. PRELIMINARIES

2.1. Properties of the ground state. Weinstein in [17] proved that the sharp constant C_{GN} of Gagliardo-Nirenberg inequality for $0 < s_c < 1$

$$\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \leq C_{GN} \|\nabla u\|_{L^2(\mathbb{R}^N)}^{\frac{N(p-1)}{2}} \|u\|_{L^2(\mathbb{R}^N)}^{2 - \frac{(N-2)(p-1)}{2}} \quad (2.1)$$

is achieved by the unique minimizer $u = Q$, where Q is the ground state of

$$-(1 - s_c)Q + \Delta Q + |Q|^{p-1}Q = 0, \quad (2.2)$$

which is radial, smooth, positive, exponentially decaying at infinity. In other words, if

$$\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} = C_{GN} \|\nabla u\|_{L^2(\mathbb{R}^N)}^{\frac{N(p-1)}{2}} \|u\|_{L^2(\mathbb{R}^N)}^{2 - \frac{(N-2)(p-1)}{2}}, \quad (2.3)$$

then, there exists $\lambda_0 \in \mathbb{C}$ and $x_0 \in \mathbb{R}^N$ such that $u(x) = \lambda_0 Q(x + x_0)$.

Applying the concentration-compactness principle, the characterization of Q yields the following proposition:

Proposition 2.1. ([14]) *There exists a function $\epsilon(\rho)$, defined for small $\rho > 0$ such that $\lim_{\rho \rightarrow 0} \epsilon(\rho) = 0$, such that for all $u \in H^1$ with*

$$|||u||_{p+1} - ||Q||_{p+1}| + |||u||_2 - ||Q||_2| + |||\nabla u||_2 - ||\nabla Q||_2| \leq \rho,$$

there exist $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$ such that

$$\|u - e^{i\theta_0}Q(\cdot - x_0)\|_{H^1} \leq \epsilon(\rho).$$

Using Pohozaev identities we can get the following identities without difficulty:

$$\|Q\|_2^2 = \frac{2}{N} \|\nabla Q\|_2^2, \quad \|Q\|_{p+1}^{p+1} = \frac{2(p+1)}{N(p-1)} \|\nabla Q\|_2^2 = \frac{(p+1)}{(p-1)} \|Q\|_2^2, \quad (2.4)$$

$$E(Q) = \frac{N(p-1)-4}{2N(p-1)} \|\nabla Q\|_2^2 = \frac{N(p-1)-4}{4(p-1)} \|Q\|_2^2 = \frac{N(p-1)-4}{4(p+1)} \|Q\|_{p+1}^{p+1}, \quad (2.5)$$

and C_{GN} can be expressed by

$$C_{GN} = \frac{\|Q\|_{p+1}^{p+1}}{\|\nabla Q\|_2^{\frac{N(p-1)}{2}} \|Q\|_2^{2 - \frac{(N-2)(p-1)}{2}}}. \quad (2.6)$$

By the H^1 local theory [1], there exist $-\infty \leq T_- < 0 < T_+ \leq +\infty$ such that (T_-, T_+) is the maximal time interval of existence for $u(t)$ solving (1.1), and if $T_+ < +\infty$ then

$$\|\nabla u(t)\|_2 \geq \frac{C}{(T_+ - t)^{\frac{1}{p-1} - \frac{N-2}{4}}} \quad \text{as } t \uparrow T_+,$$

and a similar argument holds if $-\infty < T_-$. Moreover, as a consequence of the continuity of the flow $u(t)$, we have the following dichotomy proposition :

Proposition 2.2. *Let $u_0 \in H^1(\mathbb{R}^N)$, and let $I = (T_-, T_+)$ be the maximal time interval of existence of $u(t)$ solving (1.1) and suppose (1.6) holds.*

- (a) *If $\|u_0\|_2^{\frac{1-s_c}{s_c}} \|\nabla u_0\|_2 < \|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2$, then $I = (-\infty, +\infty)$, i.e., the solution exists globally in time, and for all time $t \in \mathbb{R}$, $\|u(t)\|_2^{\frac{1-s_c}{s_c}} \|\nabla u(t)\|_2 < \|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2$.*
- (b) *If $\|u_0\|_2^{\frac{1-s_c}{s_c}} \|\nabla u_0\|_2 = \|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2$, then $u = e^{i(1-s_c)t}Q$ up to the symmetries.*
- (c) *If $\|u_0\|_2^{\frac{1-s_c}{s_c}} \|\nabla u_0\|_2 > \|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2$, then for all $t \in I$, $\|u(t)\|_2^{\frac{1-s_c}{s_c}} \|\nabla u(t)\|_2 > \|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2$.*

Proof. By rescaling, we can assume $M(u) = M(Q)$ and $E(u) = E(Q)$. In fact, if $M(u) = \alpha M(Q)$, then we set $\lambda^{-2s_c} = \alpha$ and $\tilde{u}(x, t) = \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t)$. Thus, the assumption (1.6) implies that $M(\tilde{u}) = M(Q)$ and $E(\tilde{u}) = E(Q)$.

Case (b) is given by the variational characterization (2.3) and the uniqueness of solutions of (1.1). If Case (a) is false and suppose, by continuity, there exists t_1 such that $\|u(t_1)\|_2 = \|Q\|_2$, then by Case (b) with the initial condition at $t = t_1$, the equality holds for all times, which contradicts the condition at $t = 0$. Then Case (a) is true. We can prove Case (c) by similar arguments. \square

2.2. Properties of the linearized operator. We consider a solution u of (1.1) close to $e^{i(1-s_c)t}Q$ and write

$$u(x, t) = e^{i(1-s_c)t}(Q(x) + h(x, t)).$$

Explicitly, h satisfies that

$$i\partial_t h + \Delta h - (1 - s_c)h = -S(h), \quad (2.7)$$

where

$$S(h) \equiv |Q + h|^{p-1}(Q + h) - Q^p \equiv Vh - R(h) \quad (2.8)$$

with the linear part Vh of h defined by

$$Vh \equiv pQ^{p-1}h_1 + iQ^{p-1}h_2 \quad (2.9)$$

and $R(h) = O(Q^{p-2}|h|^2 + |h|^{p-1}h)$ with its expression:

$$R(h) \equiv Q^p + pQ^{p-1}h_1 + iQ^{p-1}h_2 - |Q + h|^{p-1}(Q + h). \quad (2.10)$$

Similar to the Strichartz's estimates associated to the classical Schrödinger operator $e^{it\Delta}$, we also have the same Strichartz inequalities as (1.2) associated to the little modified Schrödinger operator $e^{it(\Delta - (1-s_c))}$. In fact, $e^{it(\Delta - (1-s_c))}$ is no other than $e^{it(1-s_c)}e^{it\Delta}$ and

should keep the estimates (1.2). Also, one can refer to [19] for this result. Furthermore, by the expression of Vh and $R(h)$, we have the following elementary estimates: For any time interval I with $|I| < \infty$, if we set $\tilde{r} = p + 1$ and $\frac{2}{q} = N(\frac{1}{2} - \frac{1}{\tilde{r}})$, then from Hölder inequality and in view of the exponentially decay of Q at infinity, we have

$$\|Vh\|_{L^{\tilde{q}'}(I; W^{1, \tilde{r}'})} \leq C|I|^{\frac{1}{\tilde{q}'} - \frac{1}{\tilde{q}}} \|h\|_{L^{\tilde{q}}(I; W^{1, \tilde{r}})}, \quad (2.11)$$

$$\|S(h)\|_{L^{\tilde{q}'}(I; W^{1, \tilde{r}'})} \leq C|I|^{\frac{1}{\tilde{q}'} - \frac{1}{\tilde{q}}} \|h\|_{L^{\tilde{q}}(I; W^{1, \tilde{r}})} (1 + \|h\|_{L^\infty(I; H^1)}^{p-1}), \quad (2.12)$$

$$\begin{aligned} & \|R(h) - R(g)\|_{L^{\tilde{q}'}(I; L^{\tilde{r}'})} \\ & \leq C|I|^{\frac{1}{\tilde{q}'} - \frac{1}{\tilde{q}}} \|h - g\|_{L^{\tilde{q}}(I; L^{\tilde{r}})} \left(\|h\|_{L^{\tilde{q}}(I; L^{\tilde{r}})} + \|g\|_{L^{\tilde{q}}(I; L^{\tilde{r}})} + \|h\|_{L^\infty(I; H^1)}^{p-1} + \|g\|_{L^\infty(I; H^1)}^{p-1} \right) \end{aligned} \quad (2.13)$$

and

$$\begin{aligned} & \|\nabla R(h) - \nabla R(g)\|_{L^{\tilde{q}'}(I; W^{1, \tilde{r}'})} \\ & \leq C|I|^{\frac{1}{\tilde{q}'} - \frac{1}{\tilde{q}}} \|h - g\|_{L^{\tilde{q}}(I; W^{1, \tilde{r}})} \left(\|h\|_{L^{\tilde{q}}(I; W^{1, \tilde{r}})} + \|g\|_{L^{\tilde{q}}(I; W^{1, \tilde{r}})} + \|h\|_{L^\infty(I; H^1)}^{p-1} + \|g\|_{L^\infty(I; H^1)}^{p-1} \right). \end{aligned} \quad (2.14)$$

Now, let $h_1 = \operatorname{Re} h$, $h_2 = \operatorname{Im} h$. If we identify $h = h_1 + ih_2 \in \mathbb{C}$ as an element $(h_1, h_2)^T$ of \mathbb{R}^2 , then h is a solution of the equation

$$\partial_t h + \mathcal{L}h = R(h), \quad \mathcal{L} \equiv \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix}, \quad (2.15)$$

where the self-adjoint operators L_+ and L_- are defined by

$$L_+ h_1 \equiv -\Delta h_1 + (1 - s_c)h_1 - pQ^{p-1}h_1, \quad L_- h_2 \equiv -\Delta h_2 + (1 - s_c)h_2 - Q^{p-1}h_2. \quad (2.16)$$

By Weinstein [18], we have the following spectral properties of the operator \mathcal{L} :

Proposition 2.3. *Let $\sigma(\mathcal{L})$ be the spectrum of the operator \mathcal{L} defined on $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, and let $\sigma_{ess}(\mathcal{L})$ be its essential spectrum. Then*

$$\sigma_{ess}(\mathcal{L}) = \{i\xi : \xi \in \mathbb{R}, |\xi| \geq 1\}, \quad \sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, e_0\}$$

with $e_0 > 0$. Furthermore, e_0 and $-e_0$ are simple eigenvalues of \mathcal{L} with eigenfunctions $\mathcal{Y}_+, \mathcal{Y}_- = \overline{\mathcal{Y}_+} \in \mathcal{S}$, and the null-space of \mathcal{L} is spanned by the $N + 1$ vectors $\partial_{x_j} Q$, $j = 1, \dots, N$ and iQ .

By this proposition, if we let $\mathcal{Y}_1 = \operatorname{Re} \mathcal{Y}_+ = \operatorname{Re} \mathcal{Y}_-$ and $\mathcal{Y}_2 = \operatorname{Im} \mathcal{Y}_+ = -\operatorname{Im} \mathcal{Y}_-$, then

$$L_+ \mathcal{Y}_1 = e_0 \mathcal{Y}_2, \quad L_- \mathcal{Y}_2 = -e_0 \mathcal{Y}_1, \quad (2.17)$$

and the null-space of L_+ is spanned by the N vectors $\partial_{x_j} Q$, $j = 1, \dots, N$, while the null-space of L_- is spanned by Q . Moreover, also by [18], we know that the operator L_- is non-negative defined.

Define the linearized energy

$$\Phi(h) \equiv \frac{1 - s_c}{2} \int |h|^2 + \frac{1}{2} \int |\nabla h|^2 - \frac{1}{2} \int Q^{p-1} (ph_1^2 + h_2^2) = \frac{1}{2} \int (L_+ h_1) h_1 + (L_- h_2) h_2. \quad (2.18)$$

Then Φ is conserved for solutions of the linearized equation $\partial_t h + \mathcal{L}h = 0$. By explicit calculation we have

$$E(Q + h) = E(Q), \quad M(Q + h) = M(Q) \quad \Rightarrow \quad |\Phi(h)| \leq c \|h\|_{p+1}^3. \quad (2.19)$$

In fact, $M(Q + h) = M(Q)$ yields that

$$\int |h|^2 = -2 \int Q h_1. \quad (2.20)$$

On the other hand, from $E(Q + h) = E(Q)$, i.e.,

$$\frac{1}{2} \int |\nabla Q + \nabla h|^2 - \frac{1}{p+1} \int |Q + h|^{p+1} - \frac{1}{2} \int |\nabla Q|^2 + \frac{1}{p+1} \int |Q|^{p+1} = 0,$$

we obtain that

$$0 = - \int \Delta Q h_1 + \frac{1}{2} \int |\nabla h|^2 - \int Q^p h_1 - \frac{1}{2} \int Q^{p-1} (p h_1^2 + h_2^2) + O \left(\int Q^{p-2} |h|^3 \right),$$

which, combined with (2.20) and (2.18), gives (2.19) by Hölder inequalities.

We now denote by $B(g, h)$ the bilinear symmetric form associated to Φ as

$$B(g, h) = \frac{1}{2} \int (L_+ g_1) h_1 + (L_- g_2) h_2, \quad (2.21)$$

for all $g, h \in H^1$. By Proposition 2.3, for any $h \in H^1$, we have

$$B(\partial_{x_j} Q, h) = B(iQ, h) = 0. \quad (2.22)$$

Furthermore, by (2.4), we have

$$\Phi(Q) = \left(\frac{1-s_c}{2} + \frac{N}{4} + \frac{p(p+1)}{2(p-1)} \right) \|Q\|_2^2 = -\frac{p^2-1}{4(p-1)} \|Q\|_2^2 < 0. \quad (2.23)$$

Thus, (2.23) and (2.22) imply immediately that $\Phi(h) \leq 0$, for any $h \in \text{span}\{\partial_{x_j} Q, iQ, Q\}$, $j = 1, \dots, N$.

Next, we are going to find two subspaces of H^1 on which Φ is positive defined. In order to do this we consider the following orthogonality relations:

$$\int (\partial_{x_j} Q) h_1 = \int Q h_2 = 0, \quad (2.24)$$

$$\int \Delta Q h_1 = 0, \quad (2.25)$$

$$\int \mathcal{Y}_1 h_2 = \int \mathcal{Y}_2 h_1 = 0. \quad (2.26)$$

Let G_\perp be the set of $h \in H^1$ satisfying (2.24) and (2.25) and G'_\perp be the set of $h \in H^1$ satisfying (2.24) and (2.26). We then have the following:

Proposition 2.4. *There exists a constant $c > 0$ such that*

$$\Phi(h) \geq c \|h\|_{H^1}^2, \quad \forall h \in G_\perp \cap G'_\perp. \quad (2.27)$$

The idea of the proof of Proposition 2.4 follows from [18] and [9].

Proof. Firstly, when $h \in G_\perp$, we show the coercivity by two steps.

Step 1. We show $\Phi(h) \geq 0$ for $h \in H^1$ satisfying (2.25). In fact, for $u \in H^1$, let

$$I(u) = \frac{\|\nabla u\|_2^{N(p-1)/2} \|u\|_2^{2-(N-2)(p-1)/2}}{\|\nabla Q\|_2^{N(p-1)/2} \|Q\|_2^{2-(N-2)(p-1)/2}} - \frac{\|u\|_{p+1}^{p+1}}{\|Q\|_{p+1}^{p+1}}, \quad (2.28)$$

which can be shown non-negative by (2.1) and (2.2). By expansion of $I(Q + \alpha h)$ and in view of (2.25), we finally obtain that for $h \in H^1$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned} I(Q + \alpha h) &= \left(1 + \frac{N(p-1)}{4} \frac{\int |\nabla h_2|^2}{\int |\nabla Q|^2} \alpha^2\right) \left(1 + \frac{4 - (N-2)(p-1)}{2} \frac{\int Q h_1}{\int Q^2} \alpha\right. \\ &\quad \left. - \frac{4(N-2)(p-1) - (N-2)^2(p-1)^2}{16} \left(\frac{\int Q h_1}{\int Q^2}\right)^2 \alpha^2 + \frac{4 - (N-2)(p-1)}{4} \frac{\int |h|^2}{\int Q^2} \alpha^2\right) \\ &\quad - \left(1 + (p+1) \frac{\int Q^p h_1}{\int Q^{p+1}} \alpha + \frac{p+1}{2} \frac{\int Q^{p-1} (p h_1^2 + h_2^2)}{\int Q^{p+1}} \alpha^2\right) + O(\alpha^3). \end{aligned}$$

Since $I(Q) = 0$ and $I(Q + \alpha h) \geq 0$ for all real α , the linear term in α should be zero, and the quadratic term be nonnegative. Applying (2.4), we obtain finally that

$$\frac{p-1}{\|Q\|_2^2} \Phi(h) \geq \frac{4(N-2)(p-1) - (N-2)^2(p-1)^2}{16} \left(\frac{\int Q h_1}{\int Q^2}\right)^2 \geq 0.$$

Step 2. We show in this step that for h fulfils (2.24) and (2.25) there exists some $c_* > 0$ such that $\Phi(h) \geq c_* \|h\|_{H^1}^2$. We denote $\Phi(h) = \Phi_1(h_1) + \Phi_2(h_2)$ with $\Phi_1(h_1) \equiv \frac{1}{2} \int (L_+ h_1) h_1$, $\Phi_2(h_2) \equiv \frac{1}{2} \int (L_- h_2) h_2$. By step 1 and Proposition 2.3, L_+ is nonnegative on $\{\Delta Q\}^\perp$ and L_- is nonnegative. Following the arguments in [18] and [9], we first show that under the assumptions (2.24) and (2.25), there exists $c_1 > 0$ such that $\Phi_1(h_1) \geq c_1 \|h_1\|_2^2$. In fact, if not, there exists a sequence $\{f_n\}$ of H^1 such that

$$\lim_{n \rightarrow +\infty} \Phi_1(f_n) = 0, \quad \|f_n\|_2 = 1 \quad (2.29)$$

and $\int \Delta Q f_n = \int \partial_{x_j} Q f_n = 0$ for $j = 1, \dots, N$. Thus we obtain that

$$\frac{1}{2} \int |\nabla f_n|^2 = -\frac{1}{2} + \frac{p}{2} \int Q^{p-1} f_n^2 + o(1), \quad (2.30)$$

which implies that $\{f_n\}$ is bounded in H^1 . Hence, up to a subsequence, we get that there exists some $f_* \in H^1$ such that $f_n \rightharpoonup f_*$ weakly in H^1 and $\frac{p}{2} \int Q^{p-1} f_n^2 \rightarrow \frac{p}{2} \int Q^{p-1} f_*^2$. Then by (2.30), it follows that $\int Q^{p-1} f_*^2 \geq \frac{1}{p}$, and so $f_* \neq 0$. From (2.29) and the weak convergence of $\{f_n\}$, we get also $\Phi_1(f_*) \leq 0$ and $\int \Delta Q f_* = \int \partial_{x_j} Q f_* = 0$ for $j = 1, \dots, N$. $\int \Delta Q f_* = 0$, however, yields that $\Phi_1(f_*) \geq 0$ by step 1. Therefore, we obtain that

$$\Phi_1(f_*) = 0 \quad (2.31)$$

and that f_* solves the following minimization problem

$$0 = \frac{\int (L_+ f_*) f_*}{\|f_*\|_2} = \min_{f \in E \setminus \{0\}} \frac{\int (L_+ f) f}{\|f\|_2},$$

where $E \equiv \{f \in H^1 : \int \Delta Q f = \int \partial_{x_j} Q f = 0, j = 1, \dots, N\}$. Hence, there exist some Lagrange multipliers $\lambda_k, k = 0, 1, \dots, N$ such that

$$L_+ f_* = \lambda_0 \Delta Q + \lambda_j \partial_{x_j} Q, \quad j = 1, \dots, N. \quad (2.32)$$

By symmetry of Q , we get that $\int \partial_{x_j} Q \partial_{x_k} Q = 0$ for $j \neq k$ and $\int \partial_{x_j} Q \Delta Q = 0$, which together with Proposition 2.3 imply that

$$0 = - \int f_* L_+ (\partial_{x_j} Q) = \int L_+ f_* \partial_{x_j} Q = \lambda_j \int |\partial_{x_j} Q|^2,$$

showing that $\lambda_j = 0$ for $j = 1, \dots, N$. Thus,

$$L_+ f_* = \lambda_0 \Delta Q = \lambda_0 (-Q^p + (1 - s_c)Q). \quad (2.33)$$

Denote $\tilde{Q} = \frac{2}{p-1}Q + x \cdot Q$, then $\tilde{Q} = \frac{\partial}{\partial \lambda}(Q_\lambda)|_{\lambda=1}$, where $Q_\lambda \equiv \lambda^{\frac{2}{p-1}}Q(\lambda x)$. Differentiating the equality $-\Delta Q_\lambda + \lambda^2(1 - s_c)Q_\lambda - Q_\lambda^p = 0$ with respect to λ at $\lambda = 1$, we obtain that $L_+ \tilde{Q} = -2(1 - s_c)Q$. Since $L_+ Q = -(p-1)Q^p$, we obtain that

$$L_+ \left(\frac{\lambda_0}{p-1}Q - \frac{\lambda_0}{2}\tilde{Q} \right) = \lambda_0 (-Q^p + (1 - s_c)Q). \quad (2.34)$$

In view of Proposition 2.3, (2.33) and (2.34) imply that $f_* = \frac{\lambda_0}{p-1}Q - \frac{\lambda_0}{2}\tilde{Q} + \sum_{j=1}^N \mu_j \partial_{x_j} Q$ for some μ_j . Since $\int \tilde{Q} \partial_{x_j} Q = 0$ and $\int f_* \partial_{x_j} Q = 0$, we get that $\mu_j = 0$ for $j = 1, \dots, N$. Hence, $f_* = \frac{\lambda_0}{p-1}Q - \frac{\lambda_0}{2}\tilde{Q} = -\frac{\lambda_0}{2}(x \cdot \nabla Q)$. By calculation, we obtain that $\Phi_1(f_*) = -\frac{\lambda_0^2}{4} \int \Delta Q (x \cdot \nabla Q) = -\frac{\lambda_0^2}{8} \int |\nabla Q|^2$, which by (2.31) implies that $\lambda_0 = 0$ and then $f_* = 0$. This contradicts $f_* \neq 0$ obtained before. We conclude that $\Phi_1(h_1) \geq c_1 \|h_1\|_2^2$ under the assumptions (2.24) and (2.25). To complete the proof, it suffices to show that for some $c_2 > 0$,

$$\int Q h_2 = 0 \quad \Rightarrow \quad \Phi_2(h_2) \geq c_2 \|h_2\|_2^2.$$

The proof is similar as for Φ_1 and we skip it.

Now we turn to show the coercivity of Φ on G'_\perp also by two steps:

Firstly, we show that for any $h \in G'_\perp \setminus \{0\}$, $\Phi(h) > 0$. In fact, otherwise, there exists $\tilde{h} \in H^1 \setminus \{0\}$ such that

$$\int \partial_{x_j} Q \tilde{h}_1 = \int Q \tilde{h}_2 = \int \mathcal{Y}_1 \tilde{h}_2 = \int \mathcal{Y}_2 \tilde{h}_1 = 0, \quad \Phi(\tilde{h}) \leq 0, \quad j = 1, \dots, N. \quad (2.35)$$

By Proposition 2.3, $B(\partial_{x_j} Q, h) = B(iQ, h) = 0$ for any $h \in H^1$. Since, by (2.35), we also have that $B(\mathcal{Y}_+, \tilde{h}) = 0$, so we have that $\partial_{x_j} Q, iQ, \mathcal{Y}_+$ and \tilde{h} are orthogonal in the bilinear symmetric form B . Note that $\Phi(iQ) = \Phi(\partial_{x_j} Q) = \Phi(\mathcal{Y}_+) = 0$ and $\Phi(\tilde{h}) \leq 0$, then we get that for any $h \in E \equiv \text{span}\{\partial_{x_j} Q, iQ, \mathcal{Y}_+, \tilde{h}, j = 1, \dots, N\}$, $\Phi(h) \leq 0$. Following the proof

of [9], we can claim that the dimension of the set E is $N + 3$. Since we have known that Φ is definite positive on G_\perp , which is a subspace of codimension $N + 2$ of H^1 , then Φ cannot be non-positive on E with $\dim E = N + 3$. Thus we have got a contradiction, and the proof of $\Phi(h) \leq 0$ is complete.

The second step of the proof of coercivity on G'_\perp can be obtained similar to that on G_\perp by contradiction arguments and we omit the details. \square

Remark 2.5. As a consequence of Proposition 2.4, we claim that

$$\int (\Delta Q - (1 - s_c)Q) \mathcal{Y}_1 \neq 0. \quad (2.36)$$

In fact, if otherwise $\int (\Delta Q - (1 - s_c)Q) \mathcal{Y}_1 = 0$, then, by the equation (2.2), we have $\int L_+ Q \mathcal{Y}_1 = 0$, which, by (2.17), implies that $\int Q \mathcal{Y}_2 = 0$. Thus, we obtain $Q \in G'_\perp$ and, from Proposition 2.4, $\Phi(Q) > 0$, which contradicts (2.23).

3. EXISTENCE OF SPECTRAL SOLUTIONS

We construct the solutions Q^+ and Q^- of Theorem 1.5 in this section.

Proposition 3.1. *Let $A \in \mathbb{R}$. If $t_0 = t_0(A) > 0$ is large enough, then there exists a radial solution $U^A \in C^\infty([t_0, +\infty), H^\infty)$ of (1.1) such that for any $b \in \mathbb{R}$ there exists $C > 0$ such that*

$$\|U^A(t) - e^{i(1-s_c)t}Q - Ae^{(i-e_0)t}\mathcal{Y}_+\|_{H^b} \leq Ce^{-2e_0t}. \quad (3.1)$$

Remark 3.2. By (3.1),

$$\|\nabla U^A(t)\|_2^2 = \|\nabla Q\|_2^2 + 2Ae^{-e_0t} \int (\nabla Q \cdot \nabla \mathcal{Y}_1 + (1 - s_c)Q \mathcal{Y}_1) + O(e^{-2e_0t}), \quad (3.2)$$

as $t \rightarrow +\infty$. In view of (2.36), we may assume, without loss of generality, that $\nabla Q \cdot \nabla \mathcal{Y}_1 + (1 - s_c)Q \mathcal{Y}_1 > 0$, and thus, $\|\nabla U^A(t)\|_2^2 - \|\nabla Q\|_2^2$ has the sign of A for large positive time.

If we set

$$Q^+(x, t) = e^{-i(1-s_c)t_0}U^{+1}(x, t + t_0), \quad Q^-(x, t) = e^{-i(1-s_c)t_0}U^{-1}(x, t + t_0), \quad (3.3)$$

then we have got that Q^\pm satisfy the statement in Theorem 1.5 except for their behavior for the negative time, which we shall specify in Section 5 and Section 6.

3.1. Approximate solutions. First in this subsection, we restate the following proposition which is for the construction of the approximate solutions U_k^A of (1.1).

Proposition 3.3. *Let $A \in \mathbb{R}$. There exists a sequence $\{\mathcal{Z}_j^A\}_{j \geq 1} \subset \mathcal{S}$ such that $\mathcal{Z}_1^A = A\mathcal{Y}_+$ and if $k \geq 1$ and $\mathcal{V}_k^A \equiv \sum_{j=1}^k e^{-je_0t} \mathcal{Z}_j^A$, then as $t \rightarrow +\infty$*

$$\partial_t \mathcal{V}_k^A + \mathcal{L} \mathcal{V}_k^A = R(\mathcal{V}_k^A) + O(e^{-(k+1)e_0t}) \quad \text{in } \mathcal{S}. \quad (3.4)$$

Remark 3.4. Let $U_k^A \equiv e^{i(1-s_c)t}(Q + \mathcal{V}_k^A)$. Then U_k^A is an approximate solution of (1.1) which satisfies (3.1) for large t . Indeed, as $t \rightarrow +\infty$, we have

$$i\partial_t U_k^A + \Delta U_k^A + |U_k^A|^{p-1}U_k^A = O(e^{-(k+1)e_0t}) \text{ in } \mathcal{S}.$$

The proof of Proposition 3.3 is almost the same as that in [9], so we only sketch it now:

In fact, the proposition is proved by induction. Omitting the superscript A , we define first $\mathcal{Z}_1 = A\mathcal{Y}_+$ and $\mathcal{V}_1 = e^{-e_0t}\mathcal{Z}_1$, which yields (3.4) for $k = 1$. Let $\mathcal{Z}_1, \dots, \mathcal{Z}_k, k \geq 1$ are known with the corresponding \mathcal{V}_k satisfying (3.4). Expand the expression $R(\mathcal{V}_k)$ and by (3.4), there exists $\mathcal{U}_{k+1} \in \mathcal{S}$ such that

$$\partial_t \mathcal{V}_k + \mathcal{L}\mathcal{V}_k = R(\mathcal{V}_k) + e^{-(k+1)e_0t}\mathcal{U}_k + O(e^{-(k+1)e_0t}) \text{ in } \mathcal{S}.$$

By Proposition 2.3, $(k+1)e_0$ is not in the spectrum of \mathcal{L} , so we can define $\mathcal{Z}_{k+1} = -(\mathcal{L} - (k+1)e_0)^{-1}\mathcal{U}_{k+1} \in \mathcal{S}$ and $\mathcal{V}_{k+1} = \mathcal{V}_k + e^{-(k+1)e_0t}\mathcal{Z}_{k+1}$. Thus, as $t \rightarrow +\infty$,

$$\partial_t \mathcal{V}_{k+1} + \mathcal{L}\mathcal{V}_{k+1} - R(\mathcal{V}_k) = R(\mathcal{V}_k) - R(\mathcal{V}_{k+1}) + O(e^{-(k+2)e_0t}) \text{ in } \mathcal{S}.$$

Since $\mathcal{V}_j = O(e^{-e_0t})$ in \mathcal{S} for $j = k, k+1$, and $\mathcal{V}_k - \mathcal{V}_{k+1} = O(e^{-(k+1)e_0t})$, we obtain then $R(\mathcal{V}_k) - R(\mathcal{V}_{k+1}) = O(e^{-(k+2)e_0t})$ in \mathcal{S} , as $t \rightarrow +\infty$. Thus, we have obtained (3.4) for $k+1$ and complete the proof.

In the following subsections, we shall prove Proposition 3.1.

3.2. Construction of special solutions. We construct a solution U^A of (1.1) such that there exists $t_0 \in \mathbb{R}$ satisfying

$$\forall b \in \mathbb{R}, \exists C > 0 : \forall t \geq t_0, k \in \mathbb{N}, \|U^A(t) - e^{i(1-s_c)t}(Q + \mathcal{V}_k^A(t))\|_{H^b} \leq Ce^{-2e_0t} \quad (3.5)$$

with \mathcal{V}_k^A constructed in Proposition 3.3. Note that (3.5) implies (3.1), and that if we have shown it for some b_0 , it follows for $b \leq b_0$. Thus, we only consider the case $b > N/2$, since then, it is well-known that the Sobolev space H^b is a Banach algebra and we have the estimate $\|fg\|_{H^b} \leq C\|f\|_{H^b}\|g\|_{H^b}$ for any $f, g \in H^b$. In order to do this, we write

$$U^A = e^{i(1-s_c)t}(Q + h^A).$$

We are going to construct a solution of (2.15) $h^A \in C^0([t_k, +\infty), H^b)$ for k and t_k large such that

$$\|h^A(t) - \mathcal{V}_k^A(t)\|_{H^b} \leq Ce^{-(k+\frac{1}{2})e_0t}. \quad (3.6)$$

After that, we show by uniqueness argument that h^A is independent of b and k . In the sequel, we omit the superscript A for brevity.

Recall the equation (2.7) of h and define

$$\varepsilon_k(t) = i\partial_t \mathcal{V}_k + \Delta \mathcal{V}_k - (1 - s_c)\mathcal{V}_k + S(\mathcal{V}_k) \quad (3.7)$$

for $k \in \mathbb{N}$. Then, if we set $v \equiv h - \mathcal{V}_k$, from (2.7) and (3.7), we obtain that

$$i\partial_t v + \Delta v - (1 - s_c)v = -S(\mathcal{V}_k + v) + S(\mathcal{V}_k) - \varepsilon_k. \quad (3.8)$$

Note that Proposition 3.3 gives

$$\varepsilon_k(t) = O(e^{-(k+1)e_0t}). \quad (3.9)$$

We solve the corresponding integral equation

$$v(t) = \mathcal{M}(v)(t), \quad (3.10)$$

where

$$\mathcal{M}(v)(t) \equiv -i \int_t^\infty e^{i(t-s)(\Delta-(1-s_c))} \left(S(\mathcal{V}_k(s) + v(s)) - S(\mathcal{V}_k(s)) + \varepsilon_k(s) \right) ds.$$

Note that (3.6) is equivalent to $\|v(t)\|_{H^b} \leq C e^{-(k+1/2)e_0 t}$, for $t \geq t_k$. Thus, we need show that \mathcal{M} is a contraction on B , which is defined by

$$B = B(t_k, k, b) \equiv \{v \in E, \|v\|_E \leq 1\},$$

where

$$E = E(t_k, k, b) \equiv \{v \in C^0([t_k, +\infty), H^b), \|v\|_E \equiv \sup_{t \geq t_k} e^{(k+\frac{1}{2})e_0 t} \|v(t)\|_{H^b} < \infty\}.$$

Let $v \in B$. Observe that for all $t \in \mathbb{R}$, $e^{it(\Delta-(1-s_c))}$ is an isometry of H^b . By definition of S we have that

$$\|S(f) - S(g)\|_{H^b} \leq C \|f - g\|_{H^b} (1 + \|f\|_{H^b}^{p-1} + \|g\|_{H^b}^{p-1}). \quad (3.11)$$

Then, for any $t \geq t_k$,

$$\|\mathcal{M}(v)(t)\|_{H^b} \leq C \int_t^\infty \|v\|_{H^b} (1 + \|\mathcal{V}_k(s)\|_{H^b}^{p-1} + \|v(s)\|_{H^b}^{p-1}) ds + C_k \int_t^\infty e^{-(k+1)e_0 s} ds. \quad (3.12)$$

By the construction of \mathcal{V}_k , $\|\mathcal{V}_k(s)\|_{H^b} \leq C_k e^{-e_0 s}$. Moreover, since $v \in B$, $\|v(s)\|_{H^b} \leq C e^{-(k+\frac{1}{2})e_0 s}$. Hence, for any $t \geq t_k$,

$$\begin{aligned} \int_t^\infty \|v\|_{H^b} (1 + \|\mathcal{V}_k(s)\|_{H^b}^{p-1} + \|v(s)\|_{H^b}^{p-1}) ds &\leq C \int_t^\infty e^{-(k+\frac{1}{2})e_0 s} + C_k e^{-(k+\frac{1}{2}+p-1)e_0 s} ds \\ &\leq C e^{-(k+\frac{1}{2})e_0 t} \left(\frac{1}{(k+\frac{1}{2})e_0} + C_k e^{-(p-1)e_0 t} \right). \end{aligned} \quad (3.13)$$

Therefore, $\mathcal{M}(v) \in E$ and by (3.12),

$$\|\mathcal{M}(v)\|_E \leq \frac{C}{(k+\frac{1}{2})e_0} + C_k e^{-\frac{e_0}{2} t_k}.$$

Choose k large so that $\frac{C}{(k+\frac{1}{2})e_0} < \frac{1}{2}$ and then take t_k large such that $C_k e^{-\frac{e_0}{2} t_k} < \frac{1}{2}$. Then \mathcal{M} maps $B = B(t_k, k, b)$ to itself. Similarly, we can also prove that \mathcal{M} is a contraction on B .

We now show that U^A is independent of b and k . By the preceding step, for $b_0 = [\frac{N}{2}] + 1$ there exist k_0 and t_0 such that there exists a unique solution U^A of (1.1) satisfying $U^A \in C^0([t_0, \infty); H^{b_0})$ and for all $t \geq t_0$,

$$\|U^A(t) - e^{i(1-s_c)t} (Q + \mathcal{V}_{k_0}^A(t))\|_{H^{b_0}} \leq C e^{-(k_0+\frac{1}{2})e_0 t}. \quad (3.14)$$

Now, let $b_1 > b_0$, if $k_1 \geq k_0 + 1$ is large enough, there exist t_1 and $\tilde{U}^A \in C^0([t_1, \infty); H^{b_1})$ such that for all $t \geq t_0$,

$$\|\tilde{U}^A(t) - e^{i(1-s_c)t}(Q + \mathcal{V}_{k_1}^A(t))\|_{H^{b_1}} \leq Ce^{-(k_1 + \frac{1}{2})e_0 t}.$$

By the construction of \mathcal{V}_k^A ,

$$\|\mathcal{V}_{k_1}^A - \mathcal{V}_{k_0}^A\|_{H^{b_1}} \leq Ce^{-(k_0+1)e_0 t}.$$

Then, we have that

$$\|\tilde{U}^A(t) - e^{i(1-s_c)t}(Q + \mathcal{V}_{k_0}^A(t))\|_{H^{b_1}} \leq e^{-(k_1 + \frac{1}{2})e_0 t} + Ce^{-(k_0+1)e_0 t} \leq Ce^{-(k_0+1)e_0 t}. \quad (3.15)$$

In particular, \tilde{U}^A satisfies (3.14) for large t . By uniqueness in the fixed point argument $\tilde{U}^A = U^A$, and then, $U^A \in C^0([t_1, \infty); H^{b_1})$. By the persistence of regularity of (1.1), $U^A \in C^0([t_0, \infty); H^{b_1})$ and thus $U^A \in C^0([t_0, \infty); H^b)$ for any $b \in \mathbb{R}$. By the equation (1.1), we indeed show that $U^A \in C^\infty([t_0, \infty); H^b)$ for any $b \in \mathbb{R}$. Note that (3.15) implies (3.5), which conclude the proof of Proposition 3.1. \square

4. MODULATION OF THRESHOLD SOLUTIONS

For $u \in H^1$, we define

$$\delta(u) = \left| \int |\nabla Q|^2 - \int |\nabla u|^2 \right|. \quad (4.1)$$

The variational characterization of Q (Proposition 2.1) shows that if¹

$$M(u) = M(Q), \quad E(u) = E(Q), \quad (4.2)$$

and $\delta(u)$ is small enough, then there exists $\tilde{\theta}$ and \tilde{x} such that $u_{\tilde{\theta}, \tilde{x}} \equiv e^{-i\tilde{\theta}}u(\cdot + \tilde{x}) = Q + \tilde{u}$ with $\|\tilde{u}\|_{H^1} \leq \tilde{\varepsilon}(\delta(u))$, where $\tilde{\varepsilon}(\delta(u)) \rightarrow 0$ as $\delta \rightarrow 0$. Now for the solution u of equation (1.1) with small gradient variant away from Q , we aim to introduce a choice of modulation parameters σ and X for which the quantity $\delta(u)$ controls linearly $\|u_{\sigma, X} - Q\|_{\dot{H}^1}$ and other relevant parameters of the problem. The choice of parameters is made through two orthogonality conditions given by the two groups of transformations $u \mapsto e^{-i\sigma}u, \sigma \in \mathbb{R}$ and $u \mapsto u(\cdot + X), X \in \mathbb{R}^N$.

We first give a useful lemma as follows.

Lemma 4.1. *There exist $\delta_0 > 0$ and a positive function $\varepsilon(\delta)$ defined for $0 < \delta \leq \delta_0$, which tends to 0 as $\delta \rightarrow 0$ such that for all $u \in H^1$ satisfying (4.2) and $\delta(u) < \delta_0$, there exists a couple $(\sigma, X) \in \mathbb{R} \times \mathbb{R}^N$ such that $v = e^{-i\sigma}u(\cdot + X)$ satisfies*

$$\|v - Q\|_{H^1} \leq \varepsilon(\delta), \quad (4.3)$$

$$\operatorname{Im} \int Qv = 0, \quad \operatorname{Re} \int \partial_{x_k} Qv = 0, \quad k = 1, \dots, N. \quad (4.4)$$

The parameters σ and X are unique in $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^N$, and the mapping $u \mapsto (\sigma, X)$ is C^1 .

¹ Note that, by the same argument in the proof of Proposition 2.2, any solution satisfying (1.6) can be rescaled to the one satisfying (4.2).

Proof. Consider the functionals on $\mathbb{R} \times \mathbb{R}^N \times H^1$:

$$J_0 : (\sigma, X, u) \mapsto \operatorname{Im} \int e^{-i\sigma} u(x+X) Q, \quad J_k : (\sigma, X, u) \mapsto \operatorname{Re} \int e^{-i\sigma} u(x+X) \partial_k Q, \quad k = 1, \dots, N.$$

Thus, the orthogonality conditions (4.4) are equivalent to the conditions $J_j(\sigma, X, u) = 0$, $j = 0, \dots, N$. Note that $J_j(0, 0, Q) = 0$ for $j = 0, \dots, N$. By direct calculation, one can check that for $j = 0, \dots, N$ and $k = 1, \dots, N$, $\left(\frac{\partial J_j}{\partial \sigma}, \frac{\partial J_j}{\partial X_k}\right)$ is invertible at $(0, 0, Q)$. By the Implicit Function Theorem, there exist $\epsilon_0, \eta_0 > 0$ such that for $u \in H^1$ satisfying $\|u - Q\|_{H^1} < \epsilon_0$, there exists $(\sigma, X) \in \mathbb{R} \times \mathbb{R}^N$ with $|\sigma| + |X| \leq \eta_0$ such that $J_j(\sigma, X, Q) = 0$. Now for $u \in H^1$ satisfying (4.2) and $\delta(u) < \delta_0$, by Proposition 2.1, we can choose $\tilde{\theta}$ and \tilde{X} such that $e^{-i\tilde{\theta}} u(\cdot + \tilde{X})$ is close to Q in H^1 , and so, as argued above, get $(\sigma, X) \in \mathbb{R} \times \mathbb{R}^N$ required in the lemma. Also by the Implicit Function Theorem, we can show the uniqueness of (σ, X) and the regularity of the mapping $u \mapsto (\sigma, X)$, concluding the proof. \square

Let u be a solution of (1.1) satisfying (4.2). For convenience, we write $\delta(t) \equiv \delta(u(t))$ and set $D_{\delta_0} \equiv \{t : \delta(t) < \delta_0\}$. By Lemma 4.1, we can define functions $\sigma(t), X(t) \in C^1$ on D_{δ_0} . Using the modulation theory to do some perturbative analysis, we write

$$e^{-i\theta(t)-i(1-s_c)t} u(t, x + X(t)) = (1 + \alpha(t))Q(x) + h(t, x), \quad (4.5)$$

with

$$\alpha(t) = \operatorname{Re} \frac{e^{-i\theta(t)-i(1-s_c)t} \int \nabla u(t, x + X(t)) \cdot \nabla Q(x)}{\int |\nabla Q|^2} - 1.$$

In fact, we choose α like this such that h satisfies the orthogonality condition (2.25).

Lemma 4.2. *Let the solution u of (1.1) satisfy (4.2). Taking δ_0 small if necessary, the following estimate hold for $t \in D_{\delta_0}$:*

$$|\alpha(t)| \approx \left| \int Q h_1(t) \right| \approx \|h(t)\|_{H^1} \approx \delta(t). \quad (4.6)$$

Proof. Let $\tilde{\delta}(t) \equiv |\alpha(t)| + \delta(t) + \|h(t)\|_{H^1}$. By Lemma 4.1, we know that $\tilde{\delta}(t)$ is small when $\delta(t)$ is small. From the equalities $M(Q + \alpha Q + h) = M(u) = M(Q)$ we obtain $\int |\alpha Q + h|^2 + 2\alpha \int Q^2 + 2 \int Q h_1$, which implies then

$$|\alpha(t)| = \frac{1}{M(Q)} \left| \int Q h_1(t) \right| + O(\tilde{\delta}^2). \quad (4.7)$$

By the orthogonality condition (2.25), we get

$$\delta(t) = \left| \int |\nabla(Q + \alpha Q + h)|^2 - \int |\nabla Q|^2 \right| = \left| (2\alpha + \alpha^2) \int |\nabla Q|^2 + \int |\nabla h|^2 \right|,$$

which implies

$$|\alpha(t)| = \frac{1}{2\|\nabla Q\|_2^2} \delta + O(\tilde{\delta}^2). \quad (4.8)$$

The orthogonality condition $\int \nabla Q \cdot \nabla h_1 = 0$ together with the equation (2.2) implies that $\int Q^p h_1 = (1-s_c) \int Q h_1$. Thus, $B(Q, h) = -\frac{1}{2}(p-1)(1-s_c) \int Q h_1 = -(1-\frac{(N-2)(p-1)}{4}) \int Q h_1$. This combined with (2.19) gives

$$\left| \alpha^2 \Phi(Q) + \Phi(h) - 2\alpha \int Q h_1 \right| = |\Phi(\alpha Q + h)| = O(\alpha^3 + \|h\|_{H^1}^3).$$

So

$$\Phi(h) = \alpha^2 |\Phi(Q)| + 2\alpha \int Q h_1 + O(\alpha^3 + \|h\|_{H^1}^3). \quad (4.9)$$

On the other hand, by Proposition 2.4 and (2.19), $\Phi(h) \approx \|h\|_{H^1}^2$, which together with (4.9) implies that

$$\|h\|_{H^1} = O(|\alpha| + \left| \int Q h_1 \right| + \tilde{\delta}^{3/2}). \quad (4.10)$$

Now, (4.7) combined with (4.10) gives $\|h\|_{H^1} = O(|\alpha| + \tilde{\delta}^{3/2})$. Thus, by the definition of $\tilde{\delta}$, (4.7), (4.8) and (4.10) yields (4.6) immediately. \square

Using Lemma 4.1 and Lemma 4.2 we have the following two lemmas.

Lemma 4.3. *Under the assumption of Lemma 4.2, taking smaller δ_0 if necessary, we have for $t \in D_{\delta_0}$*

$$|\alpha'| + |X'| + |\theta'| = O(\delta). \quad (4.11)$$

Proof. Let $\delta^* = \delta(t) + |\alpha'(t)| + |X'(t)| + |\theta'(t)|$. By (4.5) and Lemma 4.2, the equation (1.1) can be rewritten as

$$i\partial_t h + \Delta h + i\alpha'Q - \theta'Q - iX' \cdot \nabla Q = O(\delta + \delta\delta^*). \quad (4.12)$$

Firstly, multiplying (4.12) by Q and integrating the real part on \mathbb{R}^N , we obtain from (2.25) that $|\theta'| = O(\delta + \delta\delta^*)$. Then by multiplying (4.12) by $\partial_{x_j} Q$, $j = 1, \dots, N$ and integrating the imaginary part, we obtain from Lemma 4.2 and $\int \Delta h \partial_{x_j} Q = O(\delta)$ that $|X'_j| = O(\delta + \delta\delta^*)$. Similarly, by multiplying (4.12) by ΔQ and integrating the imaginary part, we obtain that $|\alpha'| = O(\delta + \delta\delta^*)$. As a consequence, we obtain that $\delta^* = O(\delta + \delta\delta^*)$ which concludes our proof by choosing δ_0 small enough. \square

Lemma 4.4. *Let u be a solution of (1.1) satisfying (4.2). Assume that u is defined on $[0, +\infty)$ and that there exist c, C such that for any $t \geq 0$,*

$$\int_t^\infty \delta(s) ds \leq Ce^{-ct}. \quad (4.13)$$

Then there exist $\theta_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^N$ and $c, C > 0$ such that

$$\|u - e^{i(1-s_c)t + i\theta_0} Q(\cdot - x_0)\|_{H^1} \leq Ce^{-ct}.$$

Proof. We first announced

$$\lim_{t \rightarrow +\infty} \delta(t) = 0. \quad (4.14)$$

In fact, if not, by (4.13), there exist two increasing sequences t_n and t'_n such that $t_n < t'_n$, $\delta(t_n) \rightarrow 0$, $\delta(t'_n) = \epsilon_1$ for some $0 < \epsilon_1 < \delta_0$, and for any $t \in (t_n, t'_n)$, there holds that $0 < \delta(t) < \epsilon_1$. On $[t_n, t'_n]$, $\alpha(t)$ is well-defined. By Lemma 4.3, $|\alpha'(t)| = O(\delta(t))$, so by (4.13), $\int_{t_n}^{t'_n} |\alpha'(t)| dt \leq C e^{-ct_n}$. Hence,

$$\lim_{n \rightarrow +\infty} |\alpha(t_n) - \alpha(t'_n)| = 0. \quad (4.15)$$

By Lemma 4.2, we have $|\alpha(t)| \approx \delta(t)$. Then, the assumption $\delta(t_n) \rightarrow 0$ yields that $|\alpha(t_n)| \rightarrow 0$, which, by (4.15), implies $|\alpha(t'_n)| \rightarrow 0$ showing a contradiction with the assumption. We have shown the claim (4.14).

By (4.14), Lemma 4.2 and Lemma 4.3, we obtain that

$$\delta(t) \approx \|h(t)\|_{\dot{H}^1} \approx |\alpha(t)| = \left| - \int_t^\infty \alpha'(s) ds \right| \leq C \int_t^\infty |\alpha'(s)| ds \leq \int_t^\infty \delta(s) ds \leq C e^{-ct}. \quad (4.16)$$

Furthermore, since by Lemma 4.3, $|X'(t)| + |\theta'(t)| = O(\delta(t)) \leq C e^{-ct}$, then there exist X_∞ and θ_∞ such that

$$|X(t) - X_\infty| + |\theta(t) - \theta_\infty| \leq C e^{-ct}. \quad (4.17)$$

In view of the decomposition (4.5) of u , (4.16) and (4.17) conclude the proof of Lemma 4.4 immediately. \square

5. CONVERGENCE TO Q IN THE CASE $\|\nabla u_0\|_2 \|u_0\|_2 > \|\nabla Q\|_2 \|Q\|_2$

The goal of this section is to prove the following proposition:

Proposition 5.1. *Consider a solution u of (1.1) such that*

$$E(u) = E(Q), \quad M(u) = M(Q), \quad (5.1)$$

$$\|\nabla u_0\|_2 > \|\nabla Q\|_2, \quad (5.2)$$

which is globally defined for positive times. Assume furthermore that u_0 is either of finite variance, i.e.,

$$\int |x|^2 |u_0|^2 < +\infty, \quad (5.3)$$

or u_0 is radial. Then there exist $\theta_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^N$ and $c, C > 0$ such that

$$\|u - e^{i(1-s_c)t + i\theta_0} Q(\cdot - x_0)\|_{H^1} \leq C e^{-ct}.$$

Moreover, the negative time of existence of u is finite.

Note that Proposition 5.1 implies that the radial solution Q^+ constructed by (3.3) has finite negative time of existence.

5.1. Finite variance solutions. Proposition 5.1 in this case follows from the following lemma.

Lemma 5.2. *Let u be a solution of (1.1) satisfying (5.1), (5.2), (5.3) and*

$$T_+(u_0) = +\infty. \quad (5.4)$$

Then for all t in the interval of existence of u ,

$$\operatorname{Im} \int x \cdot \nabla u(x, t) \bar{u}(x, t) dx > 0, \quad (5.5)$$

and there exist $c, C > 0$ such that for any $t \geq 0$,

$$\int_t^\infty \delta(s) ds \leq C e^{-ct}. \quad (5.6)$$

Before proving this lemma, we first show how to use it to prove Proposition 5.1. Assuming that u is globally defined for negative times, we consider $v(x, t) = \bar{u}(x, -t)$. Thus, v is a solution of (1.1) satisfying the assumptions of Lemma 5.2. Applying (5.5) to v for all t in the domain of the existence of u , we get

$$0 < \operatorname{Im} \int x \cdot \nabla v(x, -t) \bar{v}(x, -t) dx = -\operatorname{Im} \int x \cdot \nabla u(x, t) \bar{u}(x, t) dx,$$

which contradicts (5.5). Hence, the negative time of existence of u is finite. The other assertion of Proposition 5.1 follows from (5.6) and Lemma 4.4.

Proof of Lemma 5.2:

We set $y(t) \equiv \int |x|^2 |u(x, t)|^2$. By calculation, we have that $y'(t) = 4 \operatorname{Im} \int x \cdot \nabla u \bar{u}$ and $y''(t) = 4N(p-1)E(u) - (2N(p-1) - 8) \|\nabla u\|_2^2 = 4N(p-1)E(Q) - (2N(p-1) - 8) \|\nabla u\|_2^2$. By (2.5), we get that

$$y''(t) = (2N(p-1) - 8) (\|\nabla Q\|_2^2 - \|\nabla u\|_2^2) = -(2N(p-1) - 8) \delta(t) < 0. \quad (5.7)$$

We show (5.5), which is equivalent to $y'(t) > 0$, by contradiction. If it does not hold, there exists some t_1 such that $y'(t_1) \geq 0$. Since by (5.7), $y'' < 0$, then for $t_0 > t_1$,

$$y'(t) \leq y'(t_0) < 0, \quad \forall t \geq t_0.$$

Since $T_+(u_0) = +\infty$, we obtain that $y(t) < 0$ for large t , which is a contradiction and (5.5) must hold.

We next claim that

$$(y'(t))^2 \leq C y(t) (y''(t))^2. \quad (5.8)$$

In fact, this claim follows from (5.7) and the following lemma:

Lemma 5.3. *Let $\phi \in C^1(\mathbb{R}^N)$ and $f \in H^1(\mathbb{R}^N)$. Assume that $\int |f|^2 |\nabla \phi|^2 < \infty$ and $\|f\|_2 = \|Q\|_2, E(f) = E(Q)$. Then*

$$\left| \operatorname{Im} \int (\nabla \phi \cdot \nabla f) \bar{f} \right|^2 \leq C \delta^2(f) \int |\nabla \phi|^2 |f|^2.$$

This lemma was shown in [9] for $N = 3$. Since for the general case, it is just an easy extension, we omit the proof. Taking $\phi(x) = |x|^2$ in Lemma 5.3, we get (5.8).

Now, for all t in the interval of existence of u , we have that $y'(t) > 0$ and $y''(t) < 0$ and thus,

$$\frac{y'(t)}{\sqrt{y(t)}} \leq -Cy''(t). \quad (5.9)$$

Integrating (5.9) on $[0, t]$, we get that

$$\sqrt{y(t)} - \sqrt{y(0)} \leq -C(y'(t) - y'(0)) \leq Cy'(0),$$

which shows that $y(t)$ is bounded for $t \geq 0$. Thus (5.9) gives in turn that $y'(t) \leq -Cy''(t)$, which implies then $y'(t) \leq Ce^{-ct}$. Since $y'(t) = -\int_t^\infty y''(s)ds = (2N(p-1) - 8) \int_t^\infty \delta(s)ds$, then we obtain (5.6), concluding the proof of Lemma 5.2. \square

5.2. Radial solutions. For the radial solution u of (1.1) that satisfies (5.1), (5.2) and is globally defined for positive time, we show in this subsection that u has finite variance and finish the proof of Proposition 5.1 from the finite-variance case obtained above.

Let φ be a radial function such that $0 \leq \varphi(r), \varphi''(r) \leq 2$ and that $\varphi(r) = r^2$ for $0 \leq r \leq 1$ while $\varphi(r) \equiv 0$ for $r \geq 2$. Consider the localized variance $y_R(t) = \int R^2 \varphi(\frac{x}{R}) |u(x, t)|^2 dx$. By (5.1), we compute that

$$4N(p-1)E(u) - (2N(p-1) - 8) \|\nabla u\|_2^2 = (2N(p-1) - 8) (\|\nabla Q\|_2^2 - \|\nabla u\|_2^2).$$

Since u is radial, by explicit calculation, we obtain

$$y'_R(t) = 2RI m \int \bar{u} \nabla \varphi(\frac{x}{R}) \cdot \nabla u, \quad (5.10)$$

and

$$y''_R = 4 \sum_{j,k} Re \int \partial_k \partial_j \varphi(\frac{x}{R}) \partial_k u \partial_j \bar{u} - \frac{1}{R^2} \int \Delta^2 \varphi(\frac{x}{R}) |u|^2 - \frac{2(p-1)}{p+1} \int \Delta \varphi(\frac{x}{R}) |u|^{p+1} \quad (5.11)$$

$$= (2N(p-1) - 8) (\|\nabla Q\|_2^2 - \|\nabla u\|_2^2) + A_R(u) = -(2N(p-1) - 8) \delta(t) + A_R(u),$$

where

$$\begin{aligned} A_R(u(t)) &= 4 \sum_{j \neq k} \int \partial_j \partial_k \varphi(\frac{x}{R}) \partial_j u \partial_k \bar{u} + 4 \sum_j \int \left(\partial_{x_j^2}^2 \varphi(\frac{x}{R}) - 2 \right) |\partial_j u|^2 \\ &\quad - \frac{1}{R^2} \int \Delta^2 \varphi(\frac{x}{R}) |u|^2 - \frac{2(p-1)}{p+1} \int \left(\Delta \varphi(\frac{x}{R}) - 2N \right) |u|^{p+1} \\ &= 4 \int \left(\varphi''(\frac{x}{R}) - 2 \right) |\nabla u|^2 - \frac{1}{R^2} \int \Delta^2 \varphi(\frac{x}{R}) |u|^2 - \frac{2(p-1)}{p+1} \int \left(\Delta \varphi(\frac{x}{R}) - 2N \right) |u|^{p+1}. \end{aligned} \quad (5.12)$$

We now claim that there exists $R_0 > 0$ such that for any $R > R_0$,

$$y''_R(t) \leq -(N(p-1) - 4) \delta(t). \quad (5.13)$$

By (5.12), we need to show that there exists $R_0 > 0$ such that for any $R > R_0$, $A_R(u(t)) \leq (N(p-1) - 4)\delta(t)$. In fact, we first note that, for the standing-wave solution $e^{i(1-s_c)t}Q$ of (1.1), the corresponding $y_R(t)$ is a constant and the $\delta(t)$ is identically zero, which imply that $A_R(e^{i(1-s_c)t}Q) = 0$. Now using the parameter δ_0 as in section 4, we will show the claim (5.13) in two cases.

Firstly, we assume that $t \in D_{\delta_1}$, where $\delta_1 < \delta_0$ is to be chosen later. If we denote $v \equiv \alpha Q + h$, we get from Lemma 4.2 that

$$u(t) = e^{i(1-s_c)t}(Q + v(t)), \quad \|v(t)\|_{H^1} \leq C\delta(t).$$

Noting that $\varphi''(\frac{x}{R}) - 2 = \Delta^2 \varphi(\frac{x}{R}) = \Delta \varphi(\frac{x}{R}) - 2N = 0$ for $|x| \leq R$, we obtain that

$$|A_R(u(t))| = |A_R(Q+v) - A_R(Q)| \leq C \int_{|x| \geq R} \left(Q^p |v| + |v|^{p+1} + |\nabla Q| |\nabla v| + |\nabla v|^2 + Q |v| + |v|^2 \right).$$

By the exponential decay of Q at infinity, we get that for $R > R_1 > 0$ large and δ_1 sufficiently small,

$$|A_R(u(t))| \leq C \left(e^{-cR} \delta(t) + \delta(t)^2 + \delta(t)^{p+1} \right) \leq (N(p-1) - 4)\delta(t).$$

So (5.13) holds for $R > R_1$ and $t \in D_{\delta_1}$.

Next, we fix such a δ_1 and assume that $\delta(t) \geq \delta_1$. By our assumption on φ , we know that $\int (\varphi''(\frac{x}{R}) - 2) |\nabla u|^2 \leq 0$. It suffices to bound the other two terms now. Since if $R \geq R_2 = \sqrt{\frac{CM(Q)}{\delta_1}}$,

$$\frac{1}{R^2} \int \Delta^2 \varphi(\frac{x}{R}) |u|^2 \leq \frac{C}{M}(Q) \leq \delta_1 \leq \left(\frac{N(p-1)}{2} - 2 \right) \delta(t). \quad (5.14)$$

On the other hand, from the Radial Gagliardo-Nirenberb inequality:

Lemma 5.4. [16] *For all $\delta > 0$, there exists a constant $C_\delta > 0$ such that for all $u \in \dot{H}^{s_c}$ with radial symmetry, and for all $R > 0$, we have*

$$\int_{|x| \geq R} |u|^{p+1} dx \leq \delta \int_{|x| \geq R} |\nabla u|^2 dx + \frac{C_\delta}{R^{2(1-s_c)}} \left[(\rho(u, R))^{\frac{2(p+3)}{5-p}} + (\rho(u, R))^{\frac{p+1}{2}} \right],$$

where $\rho(u, R) = \sup_{R' \geq R} \frac{1}{(R')^{2s_c}} \int_{R' \leq |x| \leq 2R'} |u|^2 dx$.

We have for all $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ and $C_Q > 0$ such that for all $u \in \dot{H}^{s_c}$ with radial symmetry and $M(u) = M(Q)$ and for all $R > 0$,

$$\int_{|x| \geq R} |u|^{p+1} dx \leq \epsilon \int_{|x| \geq R} |\nabla u|^2 dx + \frac{C_\epsilon C_Q}{R^\beta},$$

where $\beta = \min\{2 + \frac{2s_c(3p+1)}{5-p}, 2 + s_c(p-3)\} > 0$. Thus, for ϵ small and $R > R_3$ large enough,

$$C \int_{|x| \geq R} |u|^{p+1} dx \leq \epsilon(\delta(t) + \|\nabla Q\|_2^2) + \frac{C_\epsilon}{R^\beta} \leq \epsilon C_{\delta_1} \delta(t) + \frac{C_\epsilon}{R^\beta} \leq \left(\frac{N(p-1)}{2} - 2 \right) \delta(t). \quad (5.15)$$

By (5.14) and (5.15), we get the claim (5.13) in the case $\delta(t) \geq \delta_1$ also.

Next, we claim that $y'_R(t) > 0$ for all t in the interval of existence of u . In fact, if not, since $y''_R(t) < 0$ by (5.13), there must exist $t_1, \epsilon > 0$ such that for $t \geq t_1$, $y'_R(t) < -\epsilon$, which contradicts the fact that y_R is positive and u is globally defined for positive time. Thus we conclude the claim.

Since y'_R is positive and decreasing, it must have finite limit as $t \rightarrow +\infty$. Since then the integral $\int_0^\infty y''_R(t)dt < \infty$ converges, this combined with (5.13) implies that $\int_0^\infty \delta(s)ds < \infty$. Thus, there exists a subsequence $t_n \rightarrow +\infty$ such that $\delta(t_n) \rightarrow 0$. By Proposition 2.1, there exists $\theta_0 \in \mathbb{R}$ such that $u(t_n) \rightarrow e^{i\theta_0}Q$ in H^1 up to a subsequence and translation. Since $y'_R(t) > 0$, i.e., $y_R(t)$ is increasing, thus

$$y_R(0) = \int R^2 \varphi\left(\frac{x}{R}\right) |u_0|^2 \leq \int R^2 \varphi\left(\frac{x}{R}\right) |u(t_n)|^2 \leq \int R^2 \varphi\left(\frac{x}{R}\right) |Q|^2.$$

Letting $R \rightarrow +\infty$, we obtain then $\int |x|^2 |u_0|^2 < \infty$, which turn the radial case to the finite-variance one and, by the argument in Subsection 5.1, we have proved Proposition 5.1. \square

6. CONVERGENCE TO Q IN THE CASE $\|\nabla u_0\|_2 \|u_0\|_2 < \|\nabla Q\|_2 \|Q\|_2$

In this section we are to prove the following proposition and then finish the proof of Theorem 1.5.

Proposition 6.1. *Consider a solution u of (1.1) such that*

$$E(u) = E(Q), \quad M(u) = M(Q), \quad \|\nabla u_0\|_2 < \|\nabla Q\|_2, \quad (6.1)$$

which does not scatter for positive times. Then there exist $\theta_0 \in \mathbb{R}, x_0 \in \mathbb{R}^N$ and $c, C > 0$ such that

$$\|u - e^{it+i\theta_0}Q(\cdot - x_0)\|_{H^1} \leq Ce^{-ct}.$$

In subsection 6.1, we show that a solution satisfying (6.1) is compact in H^1 up to a translation $x(t)$ in space. This is a consequence, through the profile decomposition initially introduced by Keraani [12], of the scattering of subcritical solution of (1.1) shown in [20]. Then in subsection 6.2, it is shown, by a local virial identity, that the parameter $\delta(t)$ converges to 0 in mean. We conclude in subsection 6.3 the proof of Proposition 6.1 using the results obtained above. Finally, in the last subsection 6.4, we are dedicated to the behavior of Q^- for negative times, concluding the proof of Theorem 1.5.

6.1. Compactness properties.

Lemma 6.2. *Let u be a solution of (1.1) satisfying the assumptions of Proposition 6.1. Then there exists a continuous function $x(t)$ such that*

$$K \equiv \{u(x + x(t), t), t \in [0, \infty)\} \quad (6.2)$$

has a compact closure in H^1 .

We sketch the proof similar to that in [6]:

Proof. It suffices to show that for every sequence $\tau_n \geq 0$, there exists a subsequence x_n such that $u(x + x_n, \tau)$ has a limit in H^1 .

We recall the profile decomposition discussed in [6]. There exist $\psi^j \in H^1$ and sequences x_n^j, t_n^j such that

$$u(x, \tau_n) = \sum_{j=1}^M e^{-it_n^j \Delta} \psi^j(x - x_n^j) + W_n^M(x), \quad \lim_{M \rightarrow +\infty} [\lim_{n \rightarrow +\infty} \|e^{it \Delta} W_n^M\|_{S(\dot{H}^{s_c})}] = 0, \quad (6.3)$$

$$\lim_{n \rightarrow +\infty} (|t_n^j - t_n^k| + |x_n^j - x_n^k|) = +\infty. \quad (6.4)$$

For fixed M and any $0 \leq s \leq 1$, we have the asymptotic Pythagorean expansion:

$$\|\phi_n\|_{\dot{H}^s}^2 = \sum_{j=1}^M \|\psi^j\|_{\dot{H}^s}^2 + \|W_n^M\|_{\dot{H}^s}^2 + o_n(1), \quad (6.5)$$

and the energy Pythagorean decomposition

$$E(\phi_n) = \sum_{j=1}^M E(e^{-it_n^j \Delta} \psi^j) + E(W_n^M) + o_n(1). \quad (6.6)$$

We now show that there is exactly one nonzero profile. On the one hand, if for all j , $\psi^j = 0$, then u must scatter by the small data theory (Proposition 1.1) and we get a contradiction.

On the other hand, if at least two profiles are nonzero, then by the Pythagorean expansion (6.5), there exists $\epsilon > 0$ such that for all j ,

$$\|\psi^j\|_2^{\frac{1-s_c}{s_c}} \|\nabla \psi^j\|_2 \leq \|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2 - \epsilon, \quad (6.7)$$

which, by the Gagliardo-Nirenberg inequality (2.1) and (2.6), implies that $E(e^{-it_n^j \Delta} \psi^j) > 0$. Thus, by the Pythagorean expansion (6.6), we obtain also

$$M(\psi^j)^{\frac{1-s_c}{s_c}} E(e^{-it_n^j \Delta} \psi^j) \leq M(Q)^{\frac{1-s_c}{s_c}} E(Q) - \epsilon. \quad (6.8)$$

By the existence of wave operators (Proposition 1.3), there exists, for any j , a function v_0^j in H^1 such that the corresponding solution v^j of (1.1) satisfies

$$\lim_{n \rightarrow \infty} \|e^{-it_n^j \Delta} \psi^j - v^j(t_n^j)\|_{H^1} = 0.$$

Using the arguments in [20], we can show that for large M , the solution $u(x, t + \tau_n)$ of (1.1) is close to the approximate solution $u_n \equiv \sum_{j=1}^M v^j(x - x_n^j, t + t_n^j)$ for positive times. More precisely, we obtain that u_n , which is the solution of the approximate equation $i\partial_t u_n + \Delta u_n + |u_n|^{p-1} u_n = e_n$ with $e_n = |u_n|^{p-1} u_n - \sum_{j=1}^M v^j(x - x_n^j, t + t_n^j)$, satisfies the following:

- (1) For every $M > 0$, there exists $n_0 = n_0(M) \in \mathbb{N}$ such that for all $n > n_0$, $\|u_n\|_{\dot{H}^{s_c}} \leq A$ with some large A independent of M ;
- (2) For every $M, \epsilon > 0$, there exists $n_1 = n_1(M, \epsilon) \in \mathbb{N}$ such that for all $n > n_1$, $\|e_n\|_{\dot{H}^{s_c}} \leq \epsilon$;
- (3) There exists $M_1 = M_1(\epsilon)$ and $n_2 = n_2(M_1)$ sufficiently large such that for all $n > n_2$,

$$\|e^{it\Delta}(u(\tau_n) - u_n(0))\|_{\dot{H}^{s_c}} \leq \epsilon.$$

Thus, by the perturbation theory (Proposition 1.4), we obtain that $u(t + \tau_n) \approx u_n(t)$, which must scatter for positive time. This indeed yields a contradiction and so we obtain that there is only one nonzero profile.

Thus, now we have obtained that

$$u(x, \tau_n) = e^{-it_n^1 \Delta} \psi^1(x - x_n^1) + W_n^1(x), \quad \lim_{n \rightarrow +\infty} \|e^{it\Delta} W_n^1\|_{S(\dot{H}^{s_c})} = 0. \quad (6.9)$$

We also claim that

$$\lim_{n \rightarrow \infty} \|W_n^1\|_{H^1} = 0. \quad (6.10)$$

Indeed, if not, we then obtain that for some $\epsilon > 0$, $M(e^{-it_n^1 \Delta} \psi^j)^{\frac{1-s_c}{s_c}} E(e^{-it_n^1 \Delta} \psi^j) \leq M(Q)^{\frac{1-s_c}{s_c}} E(Q) - \epsilon$, which, by similar arguments as above, implies that u scatters, a contradiction.

Finally, we claim that t_n^1 is bounded and thus converges up to extracting a subsequence. Indeed, if $t_n^1 \rightarrow +\infty$, then $\|e^{it\Delta} u(\tau_n)\|_{S((-\infty, 0], \dot{H}^{s_c})} = \|e^{i(t-t_n^1)\Delta} \psi^1\|_{S((-\infty, 0], \dot{H}^{s_c})} + o_n(1) = \|e^{it\Delta} \psi^1\|_{S((-\infty, -t_n^1], \dot{H}^{s_c})} + o_n(1) \rightarrow 0$ as $n \rightarrow +\infty$. This implies that u scatters for negative time and, by Proposition 1.1, satisfies $\|u\|_{S((-\infty, \tau_n], \dot{H}^{s_c})} \rightarrow 0$ as $n \rightarrow +\infty$. Since $\tau_n > 0$, we must have $u = 0$, contradicting the assumptions. Now, if $t_n^1 \rightarrow -\infty$, $\|e^{it\Delta} u(\tau_n)\|_{S([0, +\infty), \dot{H}^{s_c})} = \|e^{it\Delta} \psi^1\|_{S([-t_n^1, +\infty), \dot{H}^{s_c})} + o_n(1) \rightarrow 0$, showing that u scatters for positive time. We get a contradiction again. Thus we have proved the claim.

Consequently, the boundedness of t_n^1 combined with (6.10) immediately implies the compactness of K . \square

Now, for the solution u of (1.1) satisfying (6.1), we have got the translation parameter $x(t)$ by Lemma 6.2. Let the parameters $X(t), \theta(t)$ and $\alpha(t)$ be defined for $t \in D_{\delta_0}$ as in Section 4. Then by (4.5) and Lemma 4.2, there exists some constant $C_0 > 0$ such that for any $t \in D_{\delta_0}$,

$$\int_{|x-X(t)| \leq 1} |\nabla u|^2 + |u|^2 \geq \int_{|x| \leq 1} |\nabla Q|^2 + |Q|^2 - C_0 \delta(t).$$

Taking δ_0 smaller if necessary, we can assume that for any $t \in D_{\delta_0}$,

$$\int_{|x+x(t)-X(t)| \leq 1} |\nabla u(x+x(t))|^2 + |u(x+x(t))|^2 \geq \epsilon_0 > 0.$$

By the compactness of \overline{K} , we know that $|x(t) - X(t)|$ is bounded on D_{δ_0} and so we can modify $x(t)$ such that

$$x(t) = X(t), \quad \forall t \in D_{\delta_0} \quad (6.11)$$

and that K defined by (6.2) remains precompact in H^1 . As was discussed in [9] and [2], it is classical that we can choose the function $x(t)$ to be continuous. As a consequence, we have shown:

Corollary 6.3. *Let u be a solution of (1.1) satisfying the assumptions of Proposition 6.1. Then with the continuous function $x(t) = X(t)$ with $X(t)$ defined by (4.5), the set K defined by (6.2) is precompact in H^1 .*

Lemma 6.4. *Let u be a solution of (1.1) satisfying the assumptions of Proposition 6.1 and $x(t)$ defined by Corollary 6.3. Then*

$$P(u) = \text{Im} \int \bar{u} \nabla u dx = 0. \quad (6.12)$$

Furthermore,

$$\lim_{t \rightarrow +\infty} \frac{x(t)}{t} = 0. \quad (6.13)$$

Proof. The proof of (6.12) is easy. Indeed, assume $P(u) \neq 0$ and consider the Galilean transformation of u i.e., $w(x, t) = e^{ix \cdot \xi_0} e^{-it|\xi_0|^2} u(x - 2\xi_0 t, t)$. As was discussed in Remark 1.7, if we take $\xi_0 = -P(u)/M(u)$ to minimize $E(w)$, then $M(w) = M(u) = M(Q)$, $E(w) < E(u) = E(Q)$ and immediately, $M(w)^{\frac{1-s_c}{s_c}} E(w) < M(Q)^{\frac{1-s_c}{s_c}} E(Q)$. By the result obtained in [20], this implies that u must scatter in H^1 , which contradicts the assumptions of the lemma concluding (6.12).

For the proof of (6.13), one can refer to [20], and there is also a similar result in [6]. \square

6.2. Convergence in mean.

Lemma 6.5. *Let u be a solution of (1.1) satisfying the assumptions of Proposition 6.1. Then*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \delta(t) dt = 0, \quad (6.14)$$

where $\delta(t)$ is defined by (4.1).

Before proving this lemma, we obtain from it the following corollary.

Corollary 6.6. *Under the assumptions of Proposition 6.1, there exists a sequence t_n with $t_n + 1 \leq t_n$ such that*

$$\lim_{n \rightarrow +\infty} \delta(t_n) = 0$$

as $t_n \rightarrow +\infty$.

Proof of Lemma 6.5:

Let $\varphi \in C^\infty$ be defined as that in subsection 5.2: $0 \leq \varphi(r), \varphi''(r) \leq 2$ and that $\varphi(r) = r^2$ for $0 \leq r \leq 1$ while $\varphi(r) \equiv 0$ for $r \geq 2$. We consider the localized variance $y_R(t) = \int R^2 \varphi(\frac{x}{R}) |u(x, t)|^2 dx$ again and recall from subsection 5.2:

$$y'_R(t) = 2R \text{Im} \int \bar{u} \nabla \varphi\left(\frac{x}{R}\right) \cdot \nabla u, \quad |y'_R(t)| \leq CR, \quad (6.15)$$

and

$$y_R'' = (2N(p-1) - 8) (\|\nabla Q\|_2^2 - \|\nabla u\|_2^2) + A_R(u) = (2N(p-1) - 8) \delta(t) + A_R(u), \quad (6.16)$$

where

$$\begin{aligned} A_R(u(t)) = & 4 \sum_{j \neq k} \int \partial_j \partial_k \varphi\left(\frac{x}{R}\right) \partial_j u \partial_k \bar{u} + 4 \sum_j \int \left(\partial_{x_j^2}^2 \varphi\left(\frac{x}{R}\right) - 2 \right) |\partial_j u|^2 \\ & - \frac{1}{R^2} \int \Delta^2 \varphi\left(\frac{x}{R}\right) |u|^2 - \frac{2(p-1)}{p+1} \int \left(\Delta \varphi\left(\frac{x}{R}\right) - 2N \right) |u|^{p+1}. \end{aligned} \quad (6.17)$$

By the properties of φ , we can obtain the estimate for $A_R(u(t))$:

$$|A_R(u(t))| \leq C \int_{|x| \geq R} |\nabla u|^2 + \frac{1}{R^2} |u|^2 + |u|^p. \quad (6.18)$$

Let $x(t) = X(t)$ be as in K defined by Corollary 6.3. By compactness of K , there exists $R_0(\epsilon) > 0$ such that

$$\int_{|x-x(t)| \geq R_0(\epsilon)} |\nabla u|^2 + |u|^2 + |u|^p \leq \epsilon, \quad \forall t \geq 0. \quad (6.19)$$

Furthermore, by (6.13), there exists $t_0(\epsilon) \geq 0$ such that

$$|x(t)| \leq \epsilon t, \quad \forall t \geq t_0(\epsilon). \quad (6.20)$$

Let $T \geq t_0(\epsilon)$ and $R = \epsilon T + R_0(\epsilon) + 1$ for $t \in [t_0(\epsilon), T]$. Since $|x(t)| \leq \epsilon T$ and $\epsilon T + R_0(\epsilon) \leq R$, we get that

$$\begin{aligned} |A_R(u(t))| & \leq C \int_{|x| \geq R} |\nabla u|^2 + \frac{1}{R^2} |u|^2 + |u|^p \\ & \leq C \int_{|x-x(t)| + |x(t)| \geq R} |\nabla u|^2 + |u|^2 + |u|^p \leq C \int_{|x-x(t)| \geq R_0(\epsilon)} |\nabla u|^2 + |u|^2 + |u|^p \leq \epsilon. \end{aligned} \quad (6.21)$$

By (6.15) and (6.16),

$$\int_{t_0(\epsilon)}^T [4\delta(t) + A_R(u(t))] dt = \int_{t_0(\epsilon)}^T y_R''(t) dt \leq |y_R'(t)| + |y_R'(t_0(\epsilon))| \leq CR.$$

(6.18) combined with (6.21) gives then, for some $C > 0$ independent of T and ϵ ,

$$\int_{t_0(\epsilon)}^T \delta(t) dt \leq C(R + T\epsilon) \leq C(R_0(\epsilon) + 1 + T\epsilon).$$

Thus, we obtain

$$\frac{1}{T} \int_0^T \delta(t) dt \leq \frac{1}{T} \int_0^{t_0(\epsilon)} \delta(t) dt + \frac{C}{T} (R_0(\epsilon) + 1) + C\epsilon.$$

Passing to the limit first as $T \rightarrow +\infty$, then letting $\epsilon \rightarrow 0$, we obtain (6.14). \square

6.3. Exponential convergence. The aim of this subsection is to prove Proposition 6.1 by using the following Lemma 6.7 which is a localized virial argument, and Lemma 6.8, a precise control of the variations of the parameter $x(t)$.

Lemma 6.7. *Let u be a solution of (1.1) satisfying the assumptions of Proposition 6.1 and $x(t)$ defined by Corollary 6.3. Then there exists a constant C such that if $0 \leq \sigma < \tau$*

$$\int_{\sigma}^{\tau} \delta(t) dt \leq C \left(1 + \sup_{\sigma \leq t \leq \tau} |x(t)| \right) (\delta(\sigma) + \delta(\tau)). \quad (6.22)$$

Proof. For $R > 0$ we consider the localized variance $y_R(t) = \int R^2 \varphi(\frac{x}{R}) |u(x, t)|^2 dx$. Recall that

$$y'_R(t) = 2R \operatorname{Im} \int \bar{u} \nabla \varphi(\frac{x}{R}) \cdot \nabla u, \quad y''_R = (2N(p-1) - 8) \delta(t) + A_R(u), \quad (6.23)$$

where $A_R(u(t))$ is defined by (6.17).

Now we show that if $\epsilon > 0$, there exists $R_{\epsilon} > 0$ such that

$$\forall t \geq 0, \quad R \geq R_{\epsilon} (|x(t)| + 1) \Rightarrow |A_R(u(t))| \leq \epsilon \delta(t). \quad (6.24)$$

The proof of the claim is divided in two cases. When $\delta(t)$ is small, we consider δ_0 as in section 4 and choose $0 < \delta_1 < \delta_0$ to be determined. For $t \in D_{\delta_1}$, let $v = h + \alpha Q$ and then from (4.5) and Lemma 4.2 we get that

$$u(x, t) = e^{i(t+\theta(t))} (Q(x - X(t)) + v(x - X(t), t)), \quad \|v\|_{H^1} \leq C \delta(t). \quad (6.25)$$

Note that fix θ_0 and X_0 , then $A_R(e^{i\theta_0} e^{it} Q(\cdot + X_0)) = 0$ for any R and t . We obtain from the definition of A_R that

$$\begin{aligned} |A_R(u)| &= |A_R(u) - A_R(e^{i\theta_0} e^{it} Q(\cdot + X_0))| \\ &\leq C \int_{|y+X(t)| \geq R} (|\nabla Q(y)| |\nabla v(y)| + |\nabla v(y)|^2 + Q(y) |v(y)| + |v(y)|^2 + |v(y)|^{p+1}) dy \\ &\leq C \int_{|y+X(t)| \geq R} e^{-|y|} (|\nabla v(y)| + |v(y)| + |v(y)|^p) dy + \int_{|y+X(t)| \geq R} (|\nabla v(y)|^2 + |v(y)|^2 + |v(y)|^{p+1}) dy, \end{aligned}$$

Since $\|v\|_{H^1} \leq C \delta(t)$ by Lemma 4.2, then choosing R_0 sufficiently large and δ_1 small enough, we obtain

$$R \geq |X(t)| + R_0, \quad \delta(t) \leq \delta_1, \quad \Rightarrow |A_R(u(t))| \geq \epsilon \delta(t). \quad (6.26)$$

Recall that by (6.11), $x(t) = X(t)$ on D_{δ_0} and (6.26) implies (6.24) for $\delta(t) < \delta_1$.

In the case $\delta(t) \geq \delta_1$, there exists some $C > 0$ such that for any $t \geq 0$,

$$\begin{aligned} |A_R(u)| &\leq C \int_{|x| \geq R} (|\nabla u(y)|^2 + |u(y)|^2 + |u(y)|^{p+1}) dx \\ &\leq C \int_{|x-x(t)| \geq R-|x(t)|} (|\nabla u(y)|^2 + |u(y)|^2 + |u(y)|^{p+1}) dx. \end{aligned}$$

By the compactness of K , there exists $R_1 > 0$ such that

$$R \geq |x(t)| + R_1, \quad \delta(t) \geq \delta_1, \quad \Rightarrow \quad |A_R(u(t))| \geq \epsilon \delta_1 \leq \epsilon \delta(t). \quad (6.27)$$

Finally, we have proved (6.24).

By (6.23) and (6.24), we obtain that there exists $R^* > 0$ such that

$$R \geq R^*(|x(t)| + 1) \quad \Rightarrow \quad y_R''(t) \geq (N(p-1) - 4)\delta(t).$$

Let $R = R^*(\sup_{\sigma \leq t \leq \tau} |x(t)| + 1)$, we obtain

$$(N(p-1) - 4) \int_{\sigma}^{\tau} \delta(t) dt \leq \int_{\sigma}^{\tau} y_R''(t) dt = y_R'(\tau) - y_R'(\sigma). \quad (6.28)$$

If $\delta(t) < \delta_0$, by (6.23) and (6.25), then

$$\begin{aligned} y_R'(t) = & 2RI m \int \bar{v}(z) \nabla \varphi\left(\frac{z + X(t)}{R}\right) \cdot \nabla Q(z) \\ & + 2RI m \int Q(z) \nabla \varphi\left(\frac{z + X(t)}{R}\right) \cdot \nabla v(z) + 2RI m \int \bar{v}(z) \nabla \varphi\left(\frac{z + X(t)}{R}\right) \cdot \nabla v(z), \end{aligned}$$

which implies by Lemma 4.2 that $|y_R'(t)| \leq CR(\delta(t) + \delta^2(t)) \leq R\delta(t)$. On the other hand, when $\delta(t) \geq \delta_0$, the above inequality follows by straightforward estimate. Hence by (6.28) and the choice $R = R^*(\sup_{\sigma \leq t \leq \tau} |x(t)| + 1)$, we obtain (6.22) and complete our proof. \square

The following lemma is to control of the variations of $x(t)$.

Lemma 6.8. *There exists a constant C such that for any $\sigma, \tau > 0$ with $\sigma + 1 \leq \tau$,*

$$|x(\tau) - x(\sigma)| \leq C \int_{\sigma}^{\tau} \delta(t) dt. \quad (6.29)$$

The proof of the lemma can be found in [9] (Lemma 6.8 there).

Now, we are ready to show Proposition 6.1.

Proof of Proposition 6.1: Consider the sequence t_n given by Corollary 6.6 and so $t_n \rightarrow +\infty$, $t_n + 1 \leq t_n$, and $\delta(t_n) \rightarrow 0$. By Lemma 6.7 and Lemma 6.8, there exists some $C_0 > 0$ such that

$$\forall n > N_0, \quad 1 + t_{N_0} \leq t_n \quad \Rightarrow \quad |x(t_{N_0}) - x(t)| \leq C_0(1 + \sup_{[t_{N_0}, t_n]} |x(t)|)[\delta(t_{N_0}) + \delta(t_n)].$$

We choose t such that $|x(t)| = \sup_{[t_{N_0}+1, t_n]} |x(s)|$ and then

$$\sup_{[t_{N_0}+1, t_n]} |x(s)| \leq C(N_0) + C_0(1 + \sup_{[t_{N_0}+1, t_n]} |x(s)|)[\delta(t_{N_0}) + \delta(t_n)]$$

with $C(N_0) = |x(N_0)| + C_0 \sup_{[t_{N_0}, t_{N_0}+1]} |x(s)|$. Fixing N_0 large enough, we can assume $\delta(t_{N_0}) + \delta(t_n) \leq 1$ and $C_0\delta(t_{N_0}) \leq \frac{1}{2}$. Thus, for $t_n \geq t_{N_0} + 1$,

$$\frac{1}{2} \sup_{[t_{N_0}+1, t_n]} |x(s)| \leq C(N_0) + \frac{1}{2} + C_0(1 + \sup_{[t_{N_0}+1, t_n]} |x(s)|)\delta(t_n).$$

Letting $n \rightarrow +\infty$, since $\delta(t_n) \rightarrow 0$, we obtain that $|x(t)|$ is bounded on $[t_{N_0} + 1, +\infty)$. By continuity, we finally obtain the boundedness of $|x(t)|$ on $[0, +\infty)$.

Lemma 6.7 combined with the boundedness of $x(t)$ gives that for any $\sigma, \tau > 0$ and $0 \leq \sigma\tau$, $\int_\sigma^\tau \delta(t)dt \leq C(\delta(\sigma) + \delta(\tau))$. If we take $\tau = t_n$ and let $n \rightarrow +\infty$, we obtain that $\int_0^\infty \delta(t)dt < \infty$. Thus, for any $\sigma > 0$, $\int_\sigma^\infty \delta(t)dt \leq C\delta(\sigma)$. By Gronwall's Lemma, we obtain that there exist $C, c > 0$

$$\int_\sigma^\infty \delta(t)dt \leq Ce^{-c\sigma}.$$

Since $\sigma > 0$ is arbitrary, we have concluded the proof of Proposition 6.1 again using Lemma 4.4. \square

6.4. Scattering of Q^- for negative times. In the final subsection, we conclude the proof of Theorem 1.5 by showing that the special solution Q^- scatters as $t \rightarrow -\infty$. If not, we apply the argument of above subsections to the solution Q^- and $\bar{Q}^-(x, -t)$ of (1.1) and obtain a parameter $x(t)$ defined for $t \in \mathbb{R}$ such that $\tilde{K} = \{Q^-(\cdot + x(t), t), t \in \mathbb{R}\}$ has a compact closure in H^1 . By the argument at the end of Subsection 6.3, $x(t)$ is bounded and $\delta(t)$ tends to 0 as $t \rightarrow \pm\infty$. A simple adjustment of Lemma 6.7 implies that if $-\infty < \sigma \leq \tau < +\infty$ then

$$\int_\sigma^\tau \delta(t)dt \leq C\left(1 + \sup_{\sigma \leq t \leq \tau} |x(t)|\right)(\delta(\sigma) + \delta(\tau)) \leq C(\delta(\sigma) + \delta(\tau)).$$

Letting $\sigma \rightarrow -\infty$ and $\tau \rightarrow +\infty$, we obtain then $\int_{\mathbb{R}} \delta(t)dt = 0$. Thus $\delta(t) = 0$ for all t which contradicts the assumption $\|\nabla u_0\|_2 < \|\nabla Q\|_2$.

7. UNIQUENESS

We will finally conclude the proof of Theorem 1.6 in this section. The main point is to show the following uniqueness result. We want to point out that our arguments in this section are different from that in [9], which are indeed invalid for our general L^2 -supercritical case.

Proposition 7.1. *Let u be a solution of (1.1) defined on $[t_0, +\infty)$ such that $E(u) = E(Q)$, $M(u) = M(Q)$. Assume that there exist $c, C > 0$ such that for any $t \geq t_0$,*

$$\|u - e^{i(1-s_c)t}Q\|_{H^1} \leq Ce^{-ct}. \quad (7.1)$$

Then there exists $A \in \mathbb{R}$ such that $u = U^A$, where U^A is defined by Proposition 3.1.

The proof of Theorem 1.6 is divided into three parts. In subsection 7.1, we analyze the linearized equation and the spectral properties of \mathcal{L} defined by (2.15), using which we conclude the proof of Proposition 7.1 in subsection 7.2. Finally, in subsection 7.3, we finish the proof of Theorem 1.6.

Throughout this section, we often use the following integral summation argument introduced in [3] (Claim 5.8 there):

Lemma 7.2. *Let $t_0 > 0$, $p \geq 1$, $a_0 \neq 0$ and E is a normed vector space. If $f \in L^p_{loc}([t_0, \infty); E)$ satisfies that*

$$\exists \tau_0 > 0, C_0 > 0, \quad \forall t \geq t_0, \quad \|f\|_{L^p([t, t+\tau_0]; E)} \leq C_0 e^{a_0 t},$$

then, for $t \geq t_0$, we have

$$\|f\|_{L^p([t, \infty); E)} \leq \frac{C_0 e^{a_0 t}}{1 - e^{a_0 \tau_0}}, \quad \text{if } a_0 < 0; \quad \|f\|_{L^p([t_0, t]; E)} \leq \frac{C_0 e^{a_0 t}}{1 - e^{-a_0 \tau_0}}, \quad \text{if } a_0 > 0.$$

7.1. Exponentially small solutions of the linearized equation. Set $\tilde{r} = p + 1$ and $\frac{2}{q} = N(\frac{1}{2} - \frac{1}{\tilde{r}})$. We consider

$$v \in C^0([t_0, +\infty), H^1), \quad g \in L^{\tilde{q}}([t_0, +\infty), W^{1, \tilde{r}})$$

such that

$$\partial_t v + \mathcal{L}v = g, \quad (x, t) \in \mathbb{R}^N \times (t_0, +\infty), \quad (7.2)$$

$$\|v(t)\|_{H^1} \leq C e^{-\gamma_1 t}, \quad \|g(t)\|_{L^{\tilde{q}}([t, +\infty), W^{1, \tilde{r}'})} \leq C e^{-\gamma_2 t}, \quad (7.3)$$

where $0 < \gamma_1 < \gamma_2$.

The following self-improving estimate is important for our analysis.

Lemma 7.3. *Under the above assumptions,*

(a) *if $\gamma_2 \leq e_0$, then $\|v(t)\|_{H^1} \leq C e^{-\gamma_2^- t}$,*

(b) *if $\gamma_2 > e_0$, then there exists $A \in \mathbb{R}$ such that $v(t) = A e^{-e_0 t} \mathcal{Y}_+ + w(t)$ with $\|w(t)\|_{H^1} \leq C e^{-\gamma_2^- t}$.*

Proof. We first recall the quadratic form Φ defined by (2.18) and the associated bilinear form B by (2.21). We have known that $B(Q_j, h) = 0$ and $\|Q_j\|_2 = 1$ for any $h \in H^1$ and $j = 0, \dots, N$, where we denote

$$Q_0 \equiv \frac{iQ}{\|Q\|_2}, \quad Q_j \equiv \frac{\partial_j Q}{\|\partial_j Q\|_2}.$$

By definition, we can obtain $\Phi(\mathcal{Y}_+) = \Phi(\mathcal{Y}_-) = 0$. Furthermore, we assert that $B(\mathcal{Y}_+, \mathcal{Y}_-) \neq 0$. In fact, if $B(\mathcal{Y}_+, \mathcal{Y}_-) = 0$, then B and Φ would be identically 0 on $\text{span}\{\partial_j Q, iQ, \mathcal{Y}_+, \mathcal{Y}_-, j = 1, \dots, N\}$ which is of dimension $N + 3$. But Φ is, by Proposition 2.4, positive on G_\perp which is of codimension $N + 2$, yielding a contradiction by Courant's min-max principle. Thus, we can normalize the eigenfunctions $\mathcal{Y}_+, \mathcal{Y}_-$ such that $B(\mathcal{Y}_+, \mathcal{Y}_-) = 1$. Then $h \in G'_\perp$ is equivalent to

$$(Q_j, h) = 0, \quad B(\mathcal{Y}_+, h) = B(\mathcal{Y}_-, h) = 0 \quad \forall j = 0, \dots, N.$$

Now we decompose $v(t)$ as

$$v(t) = \alpha_+(t) \mathcal{Y}_+ + \alpha_-(t) \mathcal{Y}_- + \sum_{j=0}^N \beta_j(t) Q_j + v_\perp(t), \quad v_\perp \in G'_\perp, \quad (7.4)$$

where

$$\begin{aligned}\beta_j(t) &= (v(t), Q_j) - \alpha_+(t)(\mathcal{Y}_+, Q_j) - \alpha_-(t)(\mathcal{Y}_-, Q_j), \\ \alpha_+(t) &= B(v(t), \mathcal{Y}_-), \quad \alpha_-(t) = B(v(t), \mathcal{Y}_+).\end{aligned}\tag{7.5}$$

Step 1. By differentiating the equation on the coefficients (7.5) and note that $B(\mathcal{L}v, v) = 0$, we obtain that

$$\frac{d}{dt}(e^{-e_0 t} \alpha_-(t)) = e^{-e_0 t} B(g, \mathcal{Y}_+), \quad \frac{d}{dt}(e^{e_0 t} \alpha_+(t)) = e^{e_0 t} B(g, \mathcal{Y}_-),\tag{7.6}$$

$$\begin{aligned}\beta'_j(t) &= (v_t - \alpha'_+ \mathcal{Y}_+ - \alpha'_- \mathcal{Y}_-, Q_j) = (g - B(g, \mathcal{Y}_-) \mathcal{Y}_+ - B(g, \mathcal{Y}_+) \mathcal{Y}_- - \mathcal{L}v, Q_j) \\ &\equiv (\tilde{v}, Q_j),\end{aligned}\tag{7.7}$$

and

$$\frac{d}{dt} \Phi(v(t)) = 2B(g, v).\tag{7.8}$$

Step 2. We now show the following estimates :

$$|\alpha_-(t)| \leq C e^{-\gamma_2 t},\tag{7.9}$$

$$|\alpha_+(t)| \leq C e^{-\gamma_2^- t}, \quad \text{if } \gamma_2 \leq e_0 \text{ or } e_0 \leq \gamma_1\tag{7.10}$$

and there exists $A \in \mathbb{R}$ such that

$$|\alpha_+(t) - A e^{-e_0 t}| \leq C e^{-\gamma_2 t}, \quad \text{if } \gamma_2 > e_0.\tag{7.11}$$

By definition (2.21),

$$\begin{aligned}2B(g, \mathcal{Y}_+) &= \int (L_+ g_1) \mathcal{Y}_1 + \int (L_- g_2) \mathcal{Y}_2 \\ &= - \int g_1 \Delta \mathcal{Y}_1 + \int (1 - s_c) g_1 \mathcal{Y}_1 - \int p Q^{p-1} g_1 \mathcal{Y}_1 - \int g_2 \Delta \mathcal{Y}_2 + \int (1 - s_c) g_2 \mathcal{Y}_2 - \int Q^{p-1} g_2 \mathcal{Y}_2\end{aligned}\tag{7.12}$$

Hence, for any time interval I with $|I| < \infty$, we have

$$\int_I |B(g, \mathcal{Y}_\pm)| dt \leq C |I|^{\frac{1}{\tilde{\sigma}}} \|g\|_{L^{\tilde{q}}(I, L^{\tilde{r}})} \|\mathcal{Y}_\pm\|_{W^{2, \tilde{r}}},$$

which, together with (7.3), implies that

$$\int_t^{t+1} |e^{-e_0 s} B(g(s), \mathcal{Y}_+)| ds \leq C e^{-e_0 t} e^{-\gamma_2 t}.$$

By Lemma 7.2, we have then

$$\int_t^\infty |e^{-e_0 s} B(g(s), \mathcal{Y}_+)| ds \leq C e^{-e_0 t} e^{-\gamma_2 t}.\tag{7.13}$$

From (7.3) we know that $e^{-e_0 t} \alpha_-(t)$ tends to 0 as t goes to infinity. Integrating the equation on α_- in (7.6) on $[t, +\infty)$, we obtain that $|\alpha_-(t)| \leq C e^{-\gamma_2 t}$ showing (7.9).

Now, we prove (7.10). In the case $e_0 < \gamma_1$, by (7.3), we have that $e^{e_0 t} \alpha_+(t)$ tends to 0 as t goes to infinity. By similar estimates as (7.13), we also have that

$$\int_t^\infty |e^{e_0 s} B(g(s), \mathcal{Y}_-)| ds \leq C e^{e_0 t} e^{-\gamma_2 t}.$$

Integrating the equation on α_+ in (7.6) on $[t, +\infty)$, we obtain that $|\alpha_+(t)| \leq C e^{-\gamma_2 t}$. In the case $\gamma_1 \leq e_0 < \gamma_2$, also by (7.3),

$$\int_t^{t+1} |e^{e_0 s} B(g(s), \mathcal{Y}_-)| ds \leq C e^{e_0 t} e^{-\gamma_2 t},$$

which together with Lemma 7.2 gives that

$$\int_{t_0}^\infty |e^{e_0 s} B(g(s), \mathcal{Y}_-)| ds \leq C e^{e_0 t_0} e^{-\gamma_2 t_0} < \infty.$$

By (7.6), $e^{e_0 t} \alpha_+(t)$ satisfies the Cauchy criterion as $t \rightarrow +\infty$. Then, there exists A such that $\lim_{t \rightarrow +\infty} e^{e_0 t} \alpha_+(t) = A$ and

$$|\alpha_+(t) - A| \leq C e^{e_0 t} e^{-\gamma_2 t},$$

showing (7.11).

In the case $\gamma_1 < \gamma_2 \leq e_0$, integrating the equation on α_+ in (7.6) on $[0, t]$, we obtain that

$$\alpha_+(t) = e^{-e_0 t} \alpha_+(0) + e^{-e_0 t} \int_0^t e^{e_0 s} B(g, \mathcal{Y}_-) ds,$$

which, by (7.3), yields that

$$\left| \int_0^t e^{e_0 s} B(g, \mathcal{Y}_-) ds \right| \leq \begin{cases} C e^{e_0 t} e^{-\gamma_2 t}, & \gamma_2 < e_0, \\ Ct, & \gamma_2 = e_0. \end{cases}$$

This shows (7.10) in this case.

In the following steps, we prove Lemma 7.3 under the conditions (7.9), (7.10) and (7.11).

Step 3. We first do with the case $\gamma_2 \leq e_0$ or $\gamma_2 > e_0$ and $A = 0$. By step 2, we have got in this case that

$$|\alpha_+(t)| + |\alpha_-(t)| \leq C e^{-\gamma_2^- t}, \quad \forall t \geq t_0. \quad (7.14)$$

Since

$$\int_t^{t+1} B(g, v) ds \leq C e^{-(\gamma_1 + \gamma_2) t},$$

we have, by Lemma 7.2, that

$$\int_t^\infty B(g, v) ds \leq C e^{-(\gamma_1 + \gamma_2) t}.$$

By (7.8) and $|\Phi(v(t))| \leq C \|v(t)\|_{H^1}^2 \rightarrow 0$ as $t \rightarrow +\infty$, we have that $|\Phi(v(t))| \leq C e^{-(\gamma_1 + \gamma_2) t}$. Note that $\Phi(v) = B(v, v) = B(v_\perp, v_\perp) + 2\alpha_+ \alpha_-$, so we obtain from Proposition 2.4 and

(7.14) that

$$\|v_\perp\|_{H^1}^2 \leq C|B(v_\perp, v_\perp)| \leq Ce^{-(\gamma_1+\gamma_2)t}. \quad (7.15)$$

Now we turn to estimate the decay of β_j . By (7.5) and the above step, we know that $|\beta_j(t)| \rightarrow 0$ as $t \rightarrow +\infty$. Moreover, by the notation of \tilde{v} ,

$$\int_t^{t+1} |(\tilde{v}, Q_j)| ds \leq C \left(e^{-\gamma_2} + \int_t^{t+1} |(\mathcal{L}v_\perp, Q_j)| ds \right) \leq C \left(e^{-\gamma_2} + \|v_\perp\|_{L^\infty H^1} \right) \leq Ce^{-(\frac{\gamma_1+\gamma_2}{2})t}.$$

Thus by (7.7) and Lemma 7.2, we obtain that

$$|\beta_j(t)| \leq Ce^{-(\frac{\gamma_1+\gamma_2}{2})t}. \quad (7.16)$$

Thus, we have got that v and g satisfy the assumption (7.3) with γ_1 replaced by $\gamma'_1 = \frac{\gamma_1+\gamma_2}{2}$. Finally, by an iteration argument, we can obtain that

$$\|v\|_{H^1} \leq Ce^{-\gamma_2^- t} \quad (7.17)$$

in the case $\gamma_2 \leq e_0$ or $\gamma_2 > e_0$ and $A = 0$.

Step 4. We finish the proof of Lemma 7.3 by dealing with the case $\gamma_2 > e_0$ and $A \neq 0$. In this case, it suffices to assume $\gamma_1 \leq e_0$ since, otherwise, we can take $A = 0$ by Step 2. Let $\tilde{v}(t) \equiv v(t) - Ae^{-e_0 t} \mathcal{Y}_+$, it holds that

$$\partial_t \tilde{v}(t) + \mathcal{L}\tilde{v}(t) = g(t), \quad \|\tilde{v}\|_{H^1} \leq Ce^{-\gamma_1 t}.$$

We consider $\tilde{\alpha}_+(t) = B(\tilde{v}(t), \mathcal{Y}_-)$, which is the corresponding coefficient of \mathcal{Y}_+ in the decomposition of \tilde{v} . By the decomposition of v , we get that $\tilde{\alpha}_+(t) = B(v(t) - Ae^{-e_0 t} \mathcal{Y}_+, \mathcal{Y}_-) = \alpha_+(t) - Ae^{-e_0 t}$. Thus by (7.11), we have that $|\tilde{\alpha}_+(t)| \leq Ce^{-\gamma_2^- t}$, turning back to the case discussed in Step 3. As a consequence,

$$\|v(t) - Ae^{-e_0 t} \mathcal{Y}_+\|_{H^1} = \|\tilde{v}(t)\|_{H^1} \leq Ce^{-\gamma_2^- t},$$

which conclude the proof of Lemma 7.3. \square

7.2. Uniqueness. We prove Proposition 7.1. For u satisfies the hypothesis, we write $u = e^{i(1-s_c)t}(Q + h)$.

Step 1. We show that if e_0^- is any positive number such that $e_0^- < e_0$, then for any $t \geq t_0$,

$$\|h(t)\|_{H^1} \leq Ce^{-e_0^- t}. \quad (7.18)$$

Indeed, from the equation (2.7), by Strichartz's estimate and (2.12), we know from the local existence theory that, for any $(q, r) \in \Lambda_0$,

$$\|h\|_{L^q([t_0, \infty); W^{1, r})} \leq C\|h(t_0)\|_{H^1} \leq Ce^{-ct}.$$

This, in turn, implies that $\|R(h)\|_{L^{\tilde{q}'}([t_0, \infty); W^{1, \tilde{r}'})} \leq Ce^{-2ct}$ satisfying the assumptions of Lemma 7.3 with $\gamma_1 = c, \gamma_2 = 2c$. If $2c > e_0$, the proof is complete; otherwise, we get by Lemma 7.3 that $\|h(t)\|_{H^1} \leq Ce^{-2c^- t}$ and then (7.18) follows by iteration arguments.

Step 2. Consider the solution U^A constructed in Proposition 3.1 and write $U^A = e^{i(1-s_c)t}(Q + h^A)$. We show that there exists $A \in \mathbb{R}$ such that for all $\gamma > 0$, there exists $C > 0$ such that for any $t \geq t_0$,

$$\|h(t) - h^A(t)\|_{H^1} \leq Ce^{-\gamma t}. \quad (7.19)$$

According to Step 1, h fulfills the assumptions of Lemma 7.3 with $\gamma_1 = e_0^-$, $\gamma_2 = 2e_0^-$. Thus, there exists $A \in \mathbb{R}$ such that

$$\|h(t) - Ae^{-e_0 t} \mathcal{Y}\|_{H^1} \leq Ce^{-2e_0^- t}. \quad (7.20)$$

By the asymptotic development of h^A obtained in Section 3,

$$\|h^A(t) - Ae^{-e_0 t} \mathcal{Y}\|_{H^1} \leq Ce^{-2e_0^- t}.$$

Thus, (7.20) implies (7.19) for any $\gamma < 2e_0$. We next show that if (7.19) holds for some $\gamma > e_0$, then it holds for $\gamma' = \gamma + \frac{1}{2}e_0$. In fact, since $h - h^A$ solves the equation

$$\partial_t(h - h^A) + \mathcal{L}(h - h^A) = R(h) - R(h^A).$$

Again from the local well-posedness theory, for any admissible pair $(q, r) \in \Lambda_0$,

$$\|h - h^A\|_{L^q([t_0, \infty); W^{1, r})} \leq C\|h(t_0) - h^A(t_0)\|_{H^1} \leq Ce^{-\gamma t},$$

which in turn gives by (2.13) that $\|R(h) - R(h^A)\|_{L^{\tilde{q}}([t_0, \infty); W^{1, \tilde{r}})} \leq Ce^{-(e_0 + \gamma)t}$. Thus, $h - h^A$ fulfills the conditions of Lemma 7.3 with $\gamma_1 = \gamma$, $\gamma_2 = \gamma + e_0$. Then we get (7.19) with γ replaced by $\gamma + \frac{1}{2}e_0$. By iteration, (7.19) holds for any $\gamma > 0$. Thus, we have obtained that $\|u - U^A\|_{H^1} \leq Ce^{-\gamma t}$ for any $\gamma > 0$ and any $t \geq t_0$. By the definition of U^A , we obtain

$$\|u - e^{i(1-s_c)t}(Q + \mathcal{V}_{k_0}^A(t))\|_{H^b} \leq Ce^{-(k_0 + \frac{1}{2})e_0 t}$$

with $\mathcal{V}_{k_0}^A$ and k_0 constructed in Proposition 3.3. Then, by the uniqueness argument in the proof of Proposition 3.1, we get then $u = U^A$, concluding Proposition 7.1.

7.3. Proof of the classification result. We finish the proof of Theorem 1.6 in this subsection. We first claim that if $A \neq 0$, $U^A = Q^+$ for $A > 0$ or $U^A = Q^-$ for $A < 0$ up to a translation in time and a multiplication by a complex number of modulus 1. Indeed, by (3.1),

$$Q^\pm(t) = e^{i(1-s_c)t}Q \pm e^{-e_0 t_0} e^{(i-e_0)t} \mathcal{Y}_+ + O(e^{-2e_0 t}) \quad \text{in } H^1. \quad (7.21)$$

Fix $A > 0$. Let $t_1 = -t_0 - \frac{1}{e_0} \log A$ such that $e^{-e_0(t_0+t_1)} = A$. By (3.1) and (7.21), we obtain that

$$e^{-it_1} Q^+(t+t_1) = e^{i(1-s_c)t}Q + e^{-e_0(t_0+t_1)} e^{(i-e_0)t} \mathcal{Y}_+ + O(e^{-2e_0 t}) = U^A + O(e^{-2e_0 t}) \quad \text{in } H^1. \quad (7.22)$$

On the other hand, $e^{-it_1} Q^+(t+t_1) - e^{i(1-s_c)t}Q \rightarrow 0$ exponentially in H^1 as $t \rightarrow +\infty$. By Proposition 7.1, there exists \tilde{A} such that $e^{-it_1} Q^+(t+t_1) = U^{\tilde{A}}$. By (7.22), we have $\tilde{A} = A$ and thus $U^A = e^{-it_1} Q^+(t+t_1)$. The case $A < 0$ can be shown similarly.

Let u satisfy the hypothesis of Theorem 1.6. We rescale u such that $E(u) = E(Q)$, $M(u) = M(Q)$.

If $\|\nabla u_0\|_2 = \|\nabla Q\|_2$, by the variational characterization of Q , $u = e^{i(1-s_c)t}Q$ up to the symmetries of the equation which yields case (b).

If $\|\nabla u_0\|_2 < \|\nabla Q\|_2$ and assume that u does not scatter for both positive and negative times. Replacing $u(x, t)$ by $\bar{u}(x, -t)$ if necessary, we may assume u does not scatter for positive times. By Proposition 6.1, there exist $\theta_0 \in \mathbb{R}, x_0 \in \mathbb{R}^N$ and $c, C > 0$ such that $\|u(t) - e^{i(1-s_c)t+i\theta_0}Q(\cdot - x_0)\|_{H^1} \leq Ce^{-ct}$ for $t > 0$. Hence, $v(x, t) = e^{-i\theta_0}u(x + x_0, t)$ satisfies the assumptions of Proposition 7.1, which shows that $v = U^A$ for some A . Since $\|\nabla u_0\|_2 < \|\nabla Q\|_2$, by Remark 3.2, the parameter A should be negative. Thus, from the arguments in the first paragraph of this subsection, we get that $v = Q^-$ up to the symmetries of the equation, yielding case (a).

We can show case (c) similar to case (a) in view of Proposition 5.1 and Proposition 7.1 and conclude the proof of Theorem 1.6. \square

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